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Conjoined games: GO-CUT and SNO-GO

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Let $\mathcal{F}$ and $\mathcal{H}$ be two impartial rulesets. We introduce the *conjoined ruleset* $(\mathcal{F} \blacktriangleright \mathcal{H})$ in which the game is played under the $\mathcal{F}$ ruleset and then, when play is no longer possible, to continue under the $\mathcal{H}$ ruleset. The games of GO-CUT and SNO-GO on a path are considered. We give nim-values for positions at the start of Phase 2 for GO-CUT, and for SNO-GO we determine the winner.

1. Introduction

The games STRATEGO, OVID'S GAME, THREE-, SIX-, and NINE-MEN'S MORRIS [3] and also BUILDING NIM [7] are examples of combinatorial games that have two phases. Specifically, in Phase 1 the board is set up and in Phase 2 the game is played. There are many other "math games" (which mathematicians want to analyze regardless of whether people actually want to play them) in which Phase 1 is not even defined, but instances of Phase 2 are analysed. For example, BOXCARS [2], END-NIM [1], PUSH [2], TOPPLING DOMINOES [8] and their variants.

In this paper, we consider playing Phase 1 as a combinatorial game as well as Phase 2 and analyze two specific games. We were introduced to this concept by Kyle Burke and Urban Larsson (personal communication).

To avoid confusion with the multiple meanings of the word "game", we refer to the *ruleset*, which describes the legal moves, and a *position*, which is an instance of the game after several (including zero) legal moves. By necessity, the position also describes the board upon which play takes place.

In an *impartial* game, both players have the same moves available.

Definition 1. Let $\mathcal{F}$ and $\mathcal{H}$ be two impartial rulesets. The *conjoined ruleset* $(\mathcal{F} \blacktriangleright \mathcal{H})$ is to play Phase 1 under the $\mathcal{F}$ ruleset and when play is no longer possible to start Phase 2 which is played under the ruleset of $\mathcal{H}$.

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Forming a conjoined game allows for an interesting Phase 1 battle before the "real" game begins. Since play in Phase 1 sets up the board, it is convenient to have the corresponding game be a placement game [5], i.e., pieces are placed but not moved or removed. The positions at the beginning of Phase 2 will have structure reflecting the Phase 1 rules, and this allows for some partial analysis. Here we indicate by GAME_I a partizan game converted into an impartial game by allowing both players to place any of the pieces. This paper explores the following two games with two-phase play:

$$\text{GO-CUT} = (\text{NOGO}_I \blacktriangleright \text{CUTTHROAT}_I), \quad \text{and} \quad \text{SNO-GO} = (\text{SNORT}_I \blacktriangleright \text{NOGO}_I).$$

Brief notes about the games: The game NOGO is known as "anti-atari go"[1], but was independently invented by Neil McKay in 2011 (see also [6]). CUTTHROAT was introduced in [9] and the full analysis when played on stars is given in [2]. SNORT is introduced in [3], Vol. 1, and is known as "CATS & DOGS" in Portugal.

Our results cover playing these conjoined games on a path, where the first player who cannot move loses. In Section 2, we obtain the values for all the possible positions of GO-CUT at the start of Phase 2 but we were not able to find the outcomes of an empty path of length n, as a function of n. By contrast, we find the outcome of an empty path for SNO-GO but were not able to find formulas for the corresponding nim-values.

It should be noted that conjoined games and the sequential compound of games [11] have similar definitions. However, the latter is formed from two positions, so that the position for the second phase is known before the game starts and the first phase only determines who plays first in the second phase. In conjoined games, the position for the second phase is determined only after the last play of the first phase.

1.1. ***Basic background.*** The basic Sprague–Grundy theory for impartial games is presented here. Readers should consult any of [3; 2; 10] for further information and proofs:

- The minimum excluded value, *mex*, of a set S is the least nonnegative integer not included in S.
- An *option* of a position H is any position that can be reached in one move.
- Recursively, the *Sprague–Grundy value*, or *nim-value*, of a position H is given by $\mathcal{G}(H) = \text{mex}\{\mathcal{G}(H') \mid H' \text{ is an option of } H\}$. Thus, if a position has no options, it has nim-value 0.
- Let H be a position; the next player to move can win if and only if $\mathcal{G}(H) > 0$.

[1] http://senseis.xmp.net/?AntiAtariGo

- Let p and q be nonnegative integers, then $p \oplus q$ signifies the *nim-sum* or *exclusive or* of p and q. It is obtained by writing p and q in binary and adding without carrying.
- The *disjunctive sum* of positions F and H, written $F + H$, is the game in which a player may move in one component but not both.
- $\mathcal{G}(F + H) = \mathcal{G}(F) \oplus \mathcal{G}(H)$.

2. GO-CUT on a path

Since we are only considering a path we will give only those rules. The generalisations to graphs is left to the reader. We leave the full analysis as an open question for the reader since the authors didn't succeed[2]. In our defence, the game has similarities to BUILDING NIM, which seems to be hard. However, Lemma 3 gives the nim-value (recall, this is denoted by $\mathcal{G}$) of a path at the end of Phase 1.

Definition 2 (GO-CUT). Initial board is a path with n vertices.

Phase 1: On a move a player chooses an uncoloured vertex ($\cdot$) and colours it either red (R) or blue (B) provided that it is contained in a subpath of red (respectively blue) vertices that has at least one end-vertex adjacent to an uncoloured vertex.

When no moves are playable under Phase 1, delete all uncoloured vertices and then delete all maximal paths which have only red vertices or only blue vertices. The game is now a disjunctive sum of components each of which contains both red and blue vertices, that is, nonmonochromatic components.

Phase 2: A player chooses a component from the disjunctive sum, deletes one of the vertices then deletes any resulting monochromatic components.

Thus, at the end of Phase 1, for example, we might have the position $[BB \cdot RBB \cdot RB \cdot R \cdot BB]$ which, after deleting the uncoloured vertices, leaves $[BB] + [RBB] + [RB] + [R] + [BB]$. Now deleting all monochromatic components, gives the starting position for Phase 2 as $[RBB] + [RB]$.

By the rules of NOGO, at the start of Phase 2, a component will consist of i blue vertices followed by j red vertices (or the reverse) for some $i, j > 0$. Call this an (i, j)-component. To extend the notation, we also refer to $(i, 0)$ and $(0, j)$ components but these correspond to empty components.

Lemma 3. *The nim-value of an* (i, j)*-component is* $((i - 1) \oplus (j - 1)) + 1$.

[2]The nim-values of GO-CUT starting on a path of n uncoloured vertices, for $n = 1$ through 15 vertices are: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 2.

Proof. Clearly, the nim-value of a $(0, j)$- or $(i, 0)$-component is 0. We will refer to an (i, j)-component by (i, j). If i and j are positive then, by induction,

$$\begin{aligned}\mathcal{G}(i, j) &= \operatorname{mex}\{\mathcal{G}(r, j), \mathcal{G}(i, s), 0 \leq r \leq i-1, 0 \leq s \leq j-1\} \\ &= \operatorname{mex}\{\{(r-1 \oplus j-1)+1, (i-1 \oplus s-1)+1, \\ &\qquad\qquad 1 \leq r \leq i-1, 1 \leq s \leq j-1\} \cup \{0\}\}.\end{aligned}$$

Note that the set $\{(r-1 \oplus j-1), (i-1 \oplus s-1), 1 \leq r \leq i-1, 1 \leq s \leq j-1\}$ is the set of nim-values for NIM played with heaps of size $i-1$ and $j-1$ and hence contains $0, 1, \ldots, (i-1 \oplus j-1)-1$ and does not contain $(i-1 \oplus j-1)$. Adding one to each value gives that both 0 and $(i-1 \oplus j-1)+1$ are missing. Since 0 is an option of (i, j), then

$$\begin{aligned}\mathcal{G}(i, j) &= \operatorname{mex}\{\mathcal{G}(r, j), \mathcal{G}(i, s), 0 \leq r \leq i-1, 0 \leq s \leq j-1\} \\ &= (i-1 \oplus j-1)+1.\end{aligned}$$

□

3. SNO-GO on a path

We were able to obtain winning strategies for the conjoined games of SNORT_I and NOGO_I on a path. We give the rules for an arbitrary graph so that a useful general tool, Lemma 6, can be introduced.

Definition 4 (SNO-GO). The initial board is a graph with n vertices.

Phase 1: On a move a player chooses an uncoloured vertex $(\cdot)$ and colours it red (R) or blue (B) provided that no red vertex is adjacent to a blue vertex.

Phase 2: When no moves are playable under Phase 1 rules, players can colour an uncoloured vertex red or blue provided that each monochromatic component has at least one vertex adjacent to an uncoloured vertex.

Thus, at the end of Phase 1, for example, we might have the position $[BB \cdot RRR \cdot BB \cdot R \cdot BB]$. At the end of the game, the position may look like $[BB \cdot RRRRBB \cdot R \cdot BB]$.

Definition 5. Let G be a graph, $x, y \in V(G)$ and $z \notin V(G)$. The *merger* of x and y results in the graph G' where $V(G') = (V(G) \setminus \{x, y\}) \cup \{z\}$, and $vw \in E(G')$ if $vw \in E(G)$ or $v = z$ and either $xw \in E(G)$ or $yw \in E(G)$.

The proof of Lemma 6 is clear and left to the reader.

Lemma 6 (reduction). *Let $\mathscr{G}$ be a* SNO-GO *position on a graph G. Suppose x, y are two adjacent vertices that are coloured the same. Let $\mathscr{G}'$ be the position on the board resulting from the merger of x, y where z has the same colour as x and all other vertices retain their colour. The nim-values of $\mathscr{G}$ and $\mathscr{G}'$ are equal.*

As a guide to intuition, consider a path with n vertices where we label the vertices $x_1, \ldots, x_n$. If two adjacent vertices are coloured the same then we can apply Lemma 6 so that each coloured subpath is reduced to size 1 after Phase 1. For example, the position $[BB \cdot RRR \cdot BB \cdot R \cdot BB]$ becomes $[B \cdot R \cdot B \cdot R \cdot B]$ after applying Lemma 6. This notion is summarised in the next Lemma.

Lemma 7. *A* SNO-GO *position on a path of n vertices at the beginning of Phase* 2 *is equal to a path of alternating coloured vertices, each separated by a single uncoloured vertex.*

Proof. At the end of Phase 1, after applying Lemma 6, all consecutive single coloured vertices (red or blue) get amalgamated into a single representative of that colour. If there is a pair of adjacent uncoloured vertices then either of them can be coloured under Phase 1 rules. Also, if x_1 or x_n is uncoloured then Phase 1 play is still possible. Hence, after all reductions, the position will consist of vertices alternating colours with a single uncoloured vertex between them and the end vertices x_1 and x_n are also coloured. □

At the end of Phase 1, we will call any uncoloured vertex a *hole*. Note that a hole will be adjacent to exactly one red and one blue vertex. We relate this game, using Lemma 6, to NODE-KAYLES [4] or equivalently to DAWSON'S CHESS [3].

Definition 8. The impartial game NODE-KAYLES is played on a graph. Players alternately choose a vertex and delete it and all adjacent vertices. The last player to move wins.

Lemma 9. *Given a path on n vertices, let $\mathcal{G}$ be a* SNO-GO *position at the start of Phase* 2 *play. Furthermore, suppose $\mathcal{G}$ has m uncoloured vertices. If $m \geq 2$ then $\mathcal{G}$ is equivalent to* NODE-KAYLES *played on a path with $m - 2$ vertices.*

Proof. It is possible that $m = 0$ or $m = 1$. In the first case, all the vertices were coloured the same. In the second, the final position is $B \cdot R$. If $m \geq 2$ playing either of the two holes at the end, without loss of generality $[B \cdot R \ldots] \to [BXR \ldots]$, $X \in \{B, R\}$, leaves an illegal Phase 2 position. Playing an interior hole, e.g., $[B \cdot R \cdot B \cdot R \cdot B \cdot R] \to [B \cdot R \cdot BXR \cdot B \cdot R]$, $X \in \{B, R\}$ eliminates playing in the two adjacent holes as legal moves. This shows that the position is now equivalent to playing NODE-KAYLES on a path of length $m - 2$. □

For ease of referencing the players, we assume that Alf plays first on the empty board and Betti plays second.

The outcome class of the sequence of NODE-KAYLES on a path is periodic with period length 34 after a preperiod of 52 and the only $\mathcal{P}$ positions are when n is even. For exact values, see the nim-value sequence for DAWSON'S CHESS in Winning Ways [3], Vol. 1. Our approach is to show that the winning player can ensure to play first at the start at Phase 2, on the equivalent of an odd NODE-KAYLES position, or win if the opponent does not allow 3 or more holes.

Before proving the Main Theorem of this section, we need the following lemmas and conventions.

We partition the path into two pieces: the *outer vertices* consisting of vertices x_1, x_2, x_{n-1}, x_n, and the *interior*, consisting of vertices $x_3, \dots, x_{n-2}$.

Lemma 10. *Let $\mathscr{G}$ be a Phase 1 SNO-GO position on a path of n vertices. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are the same colour then at the end of Phase 1 there will be an even number of holes. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are opposite colours then at the end of Phase 1 there will be an odd number of holes.*

Proof. Let $\mathscr{G}$ be a Phase 1 SNO-GO position where at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are coloured. At the end of Phase 1, positions will be as in Lemma 7. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are the same colour, then given the alternating pattern of the colours, the number of coloured vertices is odd which implies an even number of uncoloured vertices must be separating them. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ of the position are different colours, again given the alternating patterns of colours being separated by single uncoloured vertices, this implies an even number of coloured vertices, separated by an odd number of uncoloured vertices. □

Note that Lemma 11 is referring to the original path before applying Lemma 6.

Lemma 11. *Let $\mathscr{G}$ be a SNO-GO position on a path of $2k+1$ vertices at the end of Phase 1. Let h be the number of holes. If $h = 0, 2$ or $h \geq 3$ and is odd then Alf will win the game.*

Proof. If $h = 0$ then there has been an odd number of moves, the game is over and Alf had the last move.

If $h = 2$ then Phase 2 has no moves but it is Betti's turn to play and so she loses.

If $h \geq 3$ and is odd then there has been an even number of moves $(2k + 1 - h)$ in Phase 1 and thus Alf moves first in Phase 2. NODE-KAYLES on an odd number of vertices, here $h - 2$, is a first player win and so Alf can win the game. □

Theorem 12. *Consider SNO-GO played on a path of n vertices. The initial position is in $\mathcal{P}$ if n is even and in $\mathcal{N}$ if n is odd.*

Proof. First suppose n is odd.

The strategy is for Alf to colour the centre vertex, without loss of generality, blue. Now, until Betti colours an outer vertex (the first outer vertex to be coloured), Alf always plays the same move reflected across the centre vertex. When Betti finally colours an outer vertex there are now several cases to consider. Since

Betti's last move was to colour an outer vertex, without loss of generality, suppose Betti colours x_{n-1} or x_n with X.

If there are two red interior vertices then Alf colours x_1 with the opposite colour from Betti's choice. By Lemma 10, at the end of Phase 1 there will be an odd number of holes, at least 3, and by Lemma 11 Alf will win.

Thus we may suppose that at this point in play, all coloured interior vertices are blue. There are several cases to consider:

(1) Suppose every uncoloured interior vertex is adjacent to at least one blue. That is, it is now illegal to colour an interior vertex red. Alf colours x_1 with X. The number of holes will be 0 if $X =$ blue and 2 if $X =$ red. In both cases, by Lemmas 10 and 11, Alf can force a win.

(2) Suppose there are 4 interior uncoloured vertices, any pair of these are at least distance 3 apart, nor is any adjacent to a blue vertex. Alf colours x_1 with the opposite colour, hence an odd number of holes, and Alf can force at least 3 holes by having two reds separated by a blue vertex.

(3) Similarly, if there is an interior uncoloured vertex, not adjacent to any blue vertex which has an interior blue vertex between it and the closest outer vertex. By symmetry, the reflected vertex is uncoloured and not adjacent to a blue vertex. Again, Alf colours x_1 with the opposite colour, hence an odd number of holes, and Alf can force at least 3 holes by colouring one of the two uncoloured vertices red.

This leaves the situation where the outermost blue interior vertices are followed by at most 5 uncoloured interior vertices and all other uncoloured interior vertices are adjacent to a blue vertex. Since Alf will play symmetry following any Betti move between the two outer blues we can condense the centre to a single blue vertex. The positions that remain to analyze are, where it is Betti to colour one of the, say, left two vertices. The position is one of the following:

(i) $[\cdot\cdot B\cdot\cdot]$,

(ii) $[\cdot\cdot\cdot B\cdot\cdot\cdot]$,

(iii) $[\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot]$,

(iv) $[\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot]$,

(v) $[\cdot\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot\cdot]$,

(vi) $[\cdot\cdot\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot\cdot\cdot]$.

In cases (i) and (ii), Alf forces either 0 or 2 holes by playing symmetrically.

In (iii), if Betti plays x_{n-1}, again Alf forces either 0 or 2 holes by playing symmetrically. Suppose Betti plays to $[B\cdots B\cdot\cdot\cdot\cdot]$, Alf plays to $[B\cdots B\cdot\cdot B\cdot]$ forcing 0 or 2 holes. If she plays to $[R\cdots B\cdot\cdot\cdot\cdot]$, then Alf replies $[R\cdots B\cdot\cdot R\cdot]$ forcing 2 holes.

In (iv) and (v) $[\cdots\cdots B\cdots\cdots]$, Alf colours the fourth vertex from the end of the side opposite to that of Betti's last move. Alf can now force 2 or 3 holes.

In (vi), from $[\cdot X\cdots\cdots B\cdots\cdots\cdot]$ Alf plays to $[\cdot X\cdot\cdot X\cdot\cdot B\cdots\cdots\cdot]$ and can force 0 or 2 holes. From $[X\cdots\cdots B\cdots\cdots\cdot]$ Alf plays to $[X\cdots\cdots B\cdots\cdots Y]$. Suppose without loss of generality that X is red. Regardless of what Betti plays, Alf can colour a vertex red, on the other side of the centre from X. This generates an odd number (> 1) of holes.

If the board is of even length, Betti plays the reflection symmetry and a similar analysis shows that she can force a win. □

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