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Introduction to derived categories

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Derived categories were invented by Grothendieck and Verdier around 1960, not very long after the "old" homological algebra (of derived functors between abelian categories) was established. This "new" homological algebra, of derived categories and derived functors between them, provides a significantly richer and more flexible machinery than the "old" homological algebra. For instance, the important concepts of *dualizing complex* and *tilting complex* do not exist in the "old" homological algebra.

1.	The homotopy category	431
2.	The derived category	434
3.	Derived functors	435
4.	Resolutions	436
5.	DG algebras	440
6.	Commutative dualizing complexes	441
7.	Noncommutative dualizing complexes	443
8.	Tilting complexes and derived Morita theory	445
9.	Rigid dualizing complexes	447
References		450

1. The homotopy category

Suppose M is an *abelian category*. The main examples for us are these:

- A is a ring, and $\mathsf{M} = \mathsf{Mod}\, A$, the category of left A-modules.
- $(X, \mathscr{A})$ is a ringed space, and $\mathsf{M} = \mathsf{Mod}\, \mathscr{A}$, the category of sheaves of left $\mathscr{A}$-modules.

A *complex* in M is a diagram

$$M = \big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \to \cdots \big)$$

This paper is an edited version of the notes for a two-lecture minicourse given at MSRI in January 2013. Sections 1–5 are about the general theory of derived categories, and the material is taken from my manuscript "A course on derived categories" (available online). Sections 6–9 are on more specialized topics, leaning towards noncommutative algebraic geometry.

in M such that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. A morphism of complexes $\phi : M \to N$ is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & M^{-1} \arrow[r, "\mathrm{d}_M^{-1}"] \arrow[d, "\phi^{-1}"] & M^0 \arrow[r, "\mathrm{d}_M^0"] \arrow[d, "\phi^0"] & M^1 \arrow[r] \arrow[d, "\phi^1"] & \cdots \\ \cdots \arrow[r] & N^{-1} \arrow[r, "\mathrm{d}_N^{-1}"'] & N^0 \arrow[r, "\mathrm{d}_N^0"'] & N^1 \arrow[r] & \cdots \end{tikzcd} \tag{1.1}$$

in M. Let us denote by $\mathbf{C}(\mathsf{M})$ *the category of complexes* in M. It is again an abelian category; but it is also a *differential graded category*, as we now explain.

Given $M, N \in \mathbf{C}(\mathsf{M})$ we let

$$\mathrm{Hom}_{\mathsf{M}}(M, N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M^j, N^{j+i})$$

and

$$\mathrm{Hom}_{\mathsf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M, N)^i.$$

For $\phi \in \mathrm{Hom}_{\mathsf{M}}(M, N)^i$ we let

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M.$$

In this way $\mathrm{Hom}_{\mathsf{M}}(M, N)$ becomes a complex of abelian groups, i.e. a DG (differential graded) $\mathbb{Z}$-module. Given a third complex $L \in \mathbf{C}(\mathsf{M})$, composition of morphisms in M induces a homomorphism of DG $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathsf{M}}(L, M) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M, N) \to \mathrm{Hom}_{\mathsf{M}}(L, N).$$

Compare Section 5; a DG algebra is a DG category with one object.

Note that the abelian structure of $\mathbf{C}(\mathsf{M})$ can be recovered from the DG structure as follows:

$$\mathrm{Hom}_{\mathbf{C}(\mathsf{M})}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{M}}(M, N)\big),$$

the set of 0-cocycles. Indeed, for $\phi : M \to N$ of degree 0 the condition $\mathrm{d}(\phi) = 0$ is equivalent to the commutativity of the diagram (1.1).

Next we define the *homotopy category* $\mathbf{K}(\mathsf{M})$. Its objects are the complexes in M (same as $\mathbf{C}(\mathsf{M})$), and

$$\mathrm{Hom}_{\mathbf{K}(\mathsf{M})}(M, N) = \mathrm{H}^0\big(\mathrm{Hom}_{\mathsf{M}}(M, N)\big).$$

In other words, these are homotopy classes of morphisms $\phi : M \to N$ in $\mathbf{C}(\mathsf{M})$.

There is an additive functor $\mathbf{C}(\mathsf{M}) \to \mathbf{K}(\mathsf{M})$, which is the identity on objects and surjective on morphisms. The additive category $\mathbf{K}(\mathsf{M})$ is no longer abelian – it is a *triangulated category*. Let me explain what this means.

Suppose K is an additive category, with an automorphism T called the *translation* (or shift, or suspension). A *triangle* in K is a diagram of morphisms of this sort:

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L).$$

The name comes from the alternative typesetting

$$\begin{tikzcd} & N \arrow[dl, "\gamma"'] & \\ L \arrow[rr, "\alpha"] & & M \arrow[ul, "\beta"'] \end{tikzcd}$$

A triangulated category structure on K is a set of triangles called *distinguished triangles*, satisfying a list of axioms (that are not so important for us). Details can be found in [Yekutieli 2012; Schapira 2015; Hartshorne 1966; Weibel 1994; Kashiwara and Schapira 1990; Neeman 2001; Lipman and Hashimoto 2009].

The translation T of the category $\mathbf{K}(\mathsf{M})$ is defined as follows. On objects we take $\mathrm{T}(M)^i := M^{i+1}$ and $\mathrm{d}_{\mathrm{T}(M)} := -\mathrm{d}_M$. On morphisms it is $\mathrm{T}(\phi)^i := \phi^{i+1}$. For $k \in \mathbb{Z}$, the k-th translation of M is denoted by $M[k] := \mathrm{T}^k(M)$.

Given a morphism $\alpha : L \to M$ in $\mathbf{C}(\mathsf{M})$, its *cone* is the complex

$$\mathrm{cone}(\alpha) := \mathrm{T}(L) \oplus M = \begin{bmatrix} \mathrm{T}(L) \\ M \end{bmatrix}$$

with differential (in matrix notation)

$$\mathrm{d} := \begin{bmatrix} \mathrm{d}_{\mathrm{T}(L)} & 0 \\ \alpha & \mathrm{d}_M \end{bmatrix},$$

where α is viewed as a degree 1 morphism $\mathrm{T}(L) \to M$. There are canonical morphisms $M \to \mathrm{cone}(\alpha)$ and $\mathrm{cone}(\alpha) \to \mathrm{T}(L)$ in $\mathbf{C}(\mathsf{M})$.

A triangle in $\mathbf{K}(\mathsf{M})$ is distinguished if it is isomorphic, as a diagram in $\mathbf{K}(\mathsf{M})$, to the triangle

$$L \xrightarrow{\alpha} M \to \mathrm{cone}(\alpha) \to \mathrm{T}(L)$$

for some morphism $\alpha : L \to M$ in $\mathbf{C}(\mathsf{M})$. A calculation shows that $\mathbf{K}(\mathsf{M})$ is indeed triangulated (i.e. the axioms that I did not specify are satisfied).

The relation between distinguished triangles and exact sequences will be mentioned later.

Suppose K and K' are triangulated categories. A *triangulated functor* $F : \mathsf{K} \to \mathsf{K}'$ is an additive functor that commutes with the translations, and sends distinguished triangles to distinguished triangles.

Example 1.2. Let $F : \mathsf{M} \to \mathsf{M}'$ be an additive functor (not necessarily exact) between abelian categories. Extend F to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{M}')$$

in the obvious way, namely

$$\mathbf{C}(F)(M)^i := F(M^i)$$

for a complex $M = \{M^i\}_{i \in \mathbb{Z}}$. The functor $\mathbf{C}(F)$ respects homotopies, so we get an additive functor

$$\mathbf{K}(F) : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M}').$$

This is a triangulated functor.

2. The derived category

As before M is an abelian category. Given a complex $M \in \mathbf{C}(\mathsf{M})$, we can consider its cohomologies

$$\mathrm{H}^i(M) := \ker(\mathrm{d}_M^i) / \operatorname{im}(\mathrm{d}_M^{i-1}) \in \mathsf{M}.$$

Since the cohomologies are homotopy-invariant, we get additive functors

$$\mathrm{H}^i : \mathbf{K}(\mathsf{M}) \to \mathsf{M}.$$

A morphism $\psi : M \to N$ in $\mathbf{K}(\mathsf{M})$ is called a *quasi-isomorphism* if $\mathrm{H}^i(\psi)$ are isomorphisms for all i. Let us denote by $\mathbf{S}(\mathsf{M})$ the set of all quasi-isomorphisms in $\mathbf{K}(\mathsf{M})$. Clearly $\mathbf{S}(\mathsf{M})$ is a multiplicatively closed set, i.e. the composition of two quasi-isomorphisms is a quasi-isomorphism. A calculation shows that $\mathbf{S}(\mathsf{M})$ is a left and right denominator set (as in ring theory). It follows that the Ore localization $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ exists. This is an additive category, with object set

$$\operatorname{Ob}(\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}) = \operatorname{Ob}(\mathbf{K}(\mathsf{M})).$$

There is a functor

$$\mathrm{Q} : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$$

called the localization functor, which is the identity on objects. Every morphism $\chi : M \to N$ in $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ can be written as

$$\chi = \mathrm{Q}(\phi_1) \circ \mathrm{Q}(\psi_1^{-1}) = \mathrm{Q}(\psi_2^{-1}) \circ \mathrm{Q}(\phi_2)$$

for some $\phi_i \in \mathbf{K}(\mathsf{M})$ and $\psi_i \in \mathbf{S}(\mathsf{M})$.

The category $\mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}$ inherits a triangulated structure from $\mathbf{K}(\mathsf{M})$, and the localization functor Q is triangulated. There is a universal property: given a triangulated functor

$$F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$$

to a triangulated category E, such that $F(\psi)$ is an isomorphism for every ψ in $\mathbf{S}(\mathsf{M})$, there exists a unique triangulated functor

$$F_{\mathbf{S}(\mathsf{M})} : \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})} \to \mathsf{E}$$

such that

$$F_{\mathbf{S}(\mathsf{M})} \circ \mathrm{Q} = F.$$

Definition 2.1. The *derived category* of the abelian category M is the triangulated category

$$\mathbf{D}(\mathsf{M}) := \mathbf{K}(\mathsf{M})_{\mathbf{S}(\mathsf{M})}.$$

The derived category was introduced by Grothendieck and Verdier around 1960. The first published material is the book *Residues and duality*, written by Hartshorne [1966] following notes by Grothendieck.

Let $\mathbf{D}(\mathsf{M})^0$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of the complexes whose cohomology is concentrated in degree 0.

Proposition 2.2. The obvious functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ is an equivalence.

This allows us to view M as an additive subcategory of $\mathbf{D}(\mathsf{M})$. It turns out that the abelian structure of M can be recovered from this embedding.

Proposition 2.3. Consider a sequence

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

in M. This sequence is exact if and only if there is a morphism $\gamma : N \to L[1]$ in $\mathbf{D}(\mathsf{M})$ such that

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$$

is a distinguished triangle.

3. Derived functors

As before M is an abelian category. Recall the localization functor

$$\mathrm{Q} : \mathbf{K}(\mathsf{M}) \to \mathbf{D}(\mathsf{M}).$$

It is a triangulated functor, which is the identity on objects, and inverts quasi-isomorphisms.

Suppose E is some triangulated category, and $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$ a triangulated functor. We now introduce the right and left derived functors of F. These are triangulated functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E}$$

satisfying suitable universal properties.

Definition 3.1. A *right derived functor* of F is a triangulated functor

$$\mathrm{R}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : F \to \mathrm{R}F \circ \mathrm{Q}$$

of triangulated functors $\mathbf{K}(\mathsf{M}) \to \mathsf{E}$, satisfying this condition:

($*$) The pair $(\mathrm{R}F, \eta)$ is initial among all such pairs.

Being initial means that if (G, η') is another such pair, then there is a unique morphism of triangulated functors $\theta : \mathrm{R}F \to G$ s.t. $\eta' = \theta \circ \eta$. The universal condition implies that if a right derived functor $(\mathrm{R}F, \eta)$ exists, then it is unique, up to a unique isomorphism of triangulated functors.

Definition 3.2. A *left derived functor* of F is a triangulated functor

$$\mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : \mathrm{L}F \circ \mathrm{Q} \to F$$

of triangulated functors $\mathbf{K}(\mathsf{M}) \to \mathsf{E}$, satisfying this condition:

($*$) The pair $(\mathrm{L}F, \eta)$ is terminal among all such pairs.

Again, if $(\mathrm{L}F, \eta)$ exists, then it is unique up to a unique isomorphism.

There are various modifications. One of them is a contravariant triangulated functor $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$. This can be handled using the fact that $\mathbf{K}(\mathsf{M})^{\mathrm{op}}$ is triangulated, and $F : \mathbf{K}(\mathsf{M})^{\mathrm{op}} \to \mathsf{E}$ is covariant.

We will also want to derive bifunctors. Namely to a bitriangulated bifunctor

$$F : \mathbf{K}(\mathsf{M}) \times \mathbf{K}(\mathsf{M}') \to \mathsf{E}$$

we will want to associate bitriangulated bifunctors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \times \mathbf{D}(\mathsf{M}') \to \mathsf{E}.$$

This is done similarly, and I won't give details.

4. Resolutions

Consider an additive functor $F : \mathsf{M} \to \mathsf{M}'$ between abelian categories, and the corresponding triangulated functor $\mathbf{K}(F) : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{M}')$, as in Example 1.2. By slight abuse we write F instead of $\mathbf{K}(F)$. We want to construct (or prove existence) of the derived functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{M}').$$

If F is exact (namely F sends exact sequences to exact sequences), then $\mathrm{R}F = \mathrm{L}F = F$. (This is an easy exercise.) Otherwise we need *resolutions*.

The DG structure of $\mathbf{C}(\mathsf{M})$ gives, for every $M, N \in \mathbf{C}(\mathsf{M})$, a complex of abelian groups $\mathrm{Hom}_{\mathsf{M}}(M, N)$. Recall that a complex N is called acyclic if $\mathrm{H}^i(N) = 0$ for all i; i.e. N is an exact sequence in M.

Definition 4.1. (1) A complex $I \in \mathbf{K}(\mathsf{M})$ is called *K-injective* if for every acyclic $N \in \mathbf{K}(\mathsf{M})$, the complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is also acyclic.

(2) Let $M \in \mathbf{K}(\mathsf{M})$. A *K-injective resolution* of M is a quasi-isomorphism $M \to I$ in $\mathbf{K}(\mathsf{M})$, where I is K-injective.

(3) We say that $\mathbf{K}(\mathsf{M})$ *has enough K-injectives* if every $M \in \mathbf{K}(\mathsf{M})$ has some K-injective resolution.

Theorem 4.2. If $\mathbf{K}(\mathsf{M})$ has enough K-injectives, then every triangulated functor $F : \mathbf{K}(\mathsf{M}) \to \mathsf{E}$ has a right derived functor $(\mathrm{R}F, \eta)$. Moreover, for every K-injective complex $I \in \mathbf{K}(\mathsf{M})$, the morphism $\eta_I : F(I) \to \mathrm{R}F(I)$ in E is an isomorphism.

The proof/construction goes like this: for every $M \in \mathbf{K}(\mathsf{M})$ we choose a K-injective resolution $\zeta_M : M \to I_M$, and we define

$$\mathrm{R}F(M) := F(I_M)$$

and

$$\eta_M := F(\zeta_M) : F(M) \to F(I_M)$$

in E.

Regarding existence of K-injective resolutions:

Proposition 4.3. A bounded below complex of injective objects of M is a K-injective complex.

This is the type of injective resolution used in [Hartshorne 1966]. The most general statement I know is this (see [Kashiwara and Schapira 2006, Theorem 14.3.1]):

Theorem 4.4. If M is a Grothendieck abelian category, then $\mathbf{K}(\mathsf{M})$ has enough K-injectives.

This includes $\mathsf{M} = \mathsf{Mod}\, A$ for a ring A, and $\mathsf{M} = \mathsf{Mod}\, \mathscr{A}$ for a sheaf of rings $\mathscr{A}$. Actually in these cases the construction of K-injective resolutions can be done very explicitly, and it is not so difficult.

Example 4.5. Let $f : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ be a map of ringed spaces. (For instance a map of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.) The map f induces an additive functor

$$f_* : \mathsf{Mod}\, \mathscr{A}_X \to \mathsf{Mod}\, \mathscr{A}_Y$$

called push-forward, which is usually not exact (it is left exact though). Since $\mathsf{K}(\mathsf{Mod}\,\mathscr{A}_X)$ has enough K-injectives, the right derived functor

$$\mathrm{R}f_* : \mathbf{D}(\mathsf{Mod}\,\mathscr{A}_X) \to \mathbf{D}(\mathsf{Mod}\,\mathscr{A}_Y)$$

exists.

For $\mathcal{M} \in \mathsf{Mod}\,\mathscr{A}_X$ we can use an injective resolution $\mathcal{M} \to \mathcal{I}$ (in the "classical" sense), and therefore

$$\mathrm{H}^q(\mathrm{R}f_*(\mathcal{M})) \cong \mathrm{H}^q(f_*(\mathcal{I})) \cong \mathrm{R}^q f_*(\mathcal{M}),$$

where the latter is the "classical" right derived functor.

Analogously we have:

Definition 4.6. (1) A complex $P \in \mathsf{K}(\mathsf{M})$ is called *K-projective* if for every acyclic $N \in \mathsf{K}(\mathsf{M})$, the complex $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is also acyclic.

(2) Let $M \in \mathsf{K}(\mathsf{M})$. A *K-projective resolution* of M is a quasi-isomorphism $P \to M$ in $\mathsf{K}(\mathsf{M})$, where P is K-projective.

(3) We say that $\mathsf{K}(\mathsf{M})$ *has enough K-projectives* if every $M \in \mathsf{K}(\mathsf{M})$ has some K-projective resolution.

Theorem 4.7. If $\mathsf{K}(\mathsf{M})$ has enough K-projectives, then every triangulated functor $F : \mathsf{K}(\mathsf{M}) \to \mathsf{E}$ has a left derived functor $(\mathrm{L}F, \eta)$. Moreover, for every K-projective complex $P \in \mathsf{K}(\mathsf{M})$, the morphism $\eta_P : \mathrm{L}F(P) \to F(P)$ in E is an isomorphism.

The construction of $\mathrm{L}F$ is by K-projective resolutions.

Proposition 4.8. A bounded above complex of projective objects of M is a K-projective complex.

Proposition 4.9. Let A be a ring. Then $\mathsf{K}(\mathsf{Mod}\,A)$ has enough K-projectives.

The construction of K-projective resolutions in this case can be done very explicitly, and it is not difficult.

The concepts of K-injective and K-projective complexes were introduced in [Spaltenstein 1988]. They were independently discovered by others at about the same time; see [Keller 1994; Bökstedt and Neeman 1993], for example. (Some authors used the term *homotopically injective complex*.)

Example 4.10. Suppose $\mathbb{K}$ is a commutative ring and A is a $\mathbb{K}$-algebra (i.e. A is a ring and there is a homomorphism $\mathbb{K} \to \mathrm{Z}(A)$). Consider the bi-additive bifunctor

$$\mathrm{Hom}_A(-,-) : (\mathsf{Mod}\,A)^{\mathrm{op}} \times \mathsf{Mod}\,A \to \mathsf{Mod}\,\mathbb{K}.$$

We have seen how to extend this functor to complexes (this is sometimes called "product totalization"), giving rise to a bitriangulated bifunctor

$$\mathrm{Hom}_A(-,-) : \mathbf{K}(\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathbf{K}(\mathsf{Mod}\, A) \to \mathbf{K}(\mathsf{Mod}\, \mathbb{K}).$$

The right derived bifunctor

$$\mathrm{RHom}_A(-,-) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \times \mathbf{D}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, \mathbb{K})$$

can be constructed/calculated by a K-injective resolution in either the first or the second argument. Namely given $M, N \in \mathbf{K}(\mathsf{Mod}\, A)$ we can choose a K-injective resolution $N \to I$, and let

$$\mathrm{RHom}_A(M, N) := \mathrm{Hom}_A(M, I) \in \mathbf{D}(\mathsf{Mod}\, \mathbb{K}). \tag{4.11}$$

Or we can choose a K-injective resolution $M \to P$ in $\mathbf{K}(\mathsf{Mod}\, A)^{\mathrm{op}}$, which is really a K-projective resolution $P \to M$ in $\mathbf{K}(\mathsf{Mod}\, A)$, and let

$$\mathrm{RHom}_A(M, N) := \mathrm{Hom}_A(P, N) \in \mathbf{D}(\mathsf{Mod}\, \mathbb{K}). \tag{4.12}$$

The two complexes (4.11) and (4.12) are canonically related by the quasi-isomorphisms

$$\mathrm{Hom}_A(P, N) \to \mathrm{Hom}_A(P, I) \leftarrow \mathrm{Hom}_A(M, I).$$

If $M, N \in \mathsf{Mod}\, A$ then of course

$$\mathrm{H}^q\big(\mathrm{RHom}_A(M, N)\big) \cong \mathrm{Ext}^q_A(M, N),$$

where the latter is "classical" Ext.

K-projective and K-injective complexes are good also for understanding the structure of $\mathbf{D}(\mathsf{M})$.

Proposition 4.13. Suppose $P \in \mathbf{K}(\mathsf{M})$ is K-projective and $I \in \mathbf{K}(\mathsf{M})$ is K-injective. Then for any $M \in \mathbf{K}(\mathsf{M})$ the homomorphisms

$$\mathrm{Q} : \mathrm{Hom}_{\mathbf{K}(\mathsf{M})}(P, M) \to \mathrm{Hom}_{\mathbf{D}(\mathsf{M})}(P, M)$$

and

$$\mathrm{Q} : \mathrm{Hom}_{\mathbf{K}(\mathsf{M})}(M, I) \to \mathrm{Hom}_{\mathbf{D}(\mathsf{M})}(M, I)$$

are bijective.

Let us denote by $\mathbf{K}(\mathsf{M})_{\mathrm{prj}}$ and $\mathbf{K}(\mathsf{M})_{\mathrm{inj}}$ the full subcategories of $\mathbf{K}(\mathsf{M})$ on the K-projective and the K-injective complexes respectively.

Corollary 4.14. The triangulated functors

$$\mathrm{Q} : \mathbf{K}(\mathsf{M})_{\mathrm{prj}} \to \mathbf{D}(\mathsf{M})$$

and

$$\mathrm{Q} : \mathbf{K}(\mathsf{M})_{\mathrm{inj}} \to \mathbf{D}(\mathsf{M})$$

are fully faithful.

Corollary 4.15. (1) If $\mathbf{K}(\mathsf{M})$ has enough K-projectives, then the triangulated functor $\mathrm{Q} : \mathbf{K}(\mathsf{M})_{\mathrm{prj}} \to \mathbf{D}(\mathsf{M})$ is an equivalence.

(2) If $\mathbf{K}(\mathsf{M})$ has enough K-injectives, then the triangulated functor $\mathrm{Q} : \mathbf{K}(\mathsf{M})_{\mathrm{inj}} \to \mathbf{D}(\mathsf{M})$ is an equivalence.

Exercise 4.16. Let $\mathbb{K}$ be a nonzero commutative ring and $A := \mathbb{K}[t]$ the polynomial ring. We view $\mathbb{K}$ as an A-module via $t \mapsto 0$. Find a nonzero morphism $\chi : \mathbb{K} \to \mathbb{K}[1]$ in $\mathbf{D}(\mathsf{Mod}\, A)$. Show that $\mathrm{H}^q(\chi) = 0$ for all $q \in \mathbb{Z}$.

When $\mathsf{M} = \mathsf{Mod}\, A$ for a ring A, we can also talk about *K-flat* complexes. A complex P is K-flat if for any acyclic complex $N \in \mathsf{Mod}\, A^{\mathrm{op}}$ the complex $N \otimes_A P$ is acyclic. Any K-projective complex is K-flat. The left derived bifunctor $N \otimes^{\mathrm{L}}_A M$ can be constructed using K-flat resolutions of either argument:

$$N \otimes^{\mathrm{L}}_A M \cong N \otimes_A P \cong Q \otimes_A M$$

for any K-flat resolutions $P \to M$ in $\mathbf{K}(\mathsf{Mod}\, A)$ and $Q \to N$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{op}})$.

5. DG algebras

A DG algebra (or DG ring) is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, with differential d of degree 1, satisfying the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}(b)$$

for $a \in A^i$ and $b \in A^j$.

A left DG A-module is a left graded A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with differential d of degree 1, satisfying the graded Leibniz rule. Denote by $\mathsf{DGMod}\, A$ the category of left DG A-modules.

As in the ring case, for any $M, N \in \mathsf{DGMod}\, A$ there is a complex of $\mathbb{Z}$-modules $\mathrm{Hom}_A(M, N)$, and

$$\mathrm{Hom}_{\mathsf{DGMod}\, A}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_A(M, N)\big).$$

The homotopy category is $\tilde{\mathbf{K}}(\mathsf{DGMod}\, A)$, with

$$\mathrm{Hom}_{\tilde{\mathbf{K}}(\mathsf{DGMod}\, A)}(M, N) = \mathrm{H}^0\big(\mathrm{Hom}_A(M, N)\big).$$

After inverting the quasi-isomorphisms in $\tilde{\mathbf{K}}(\mathsf{DGMod}\, A)$ we obtain the derived category $\tilde{\mathbf{D}}(\mathsf{DGMod}\, A)$. These are triangulated categories.

Example 5.1. Suppose A is a ring (i.e. $A^i = 0$ for $i \neq 0$). Then $\mathsf{DGMod}\, A = \mathbf{C}(\mathsf{Mod}\, A)$ and $\tilde{\mathbf{D}}(\mathsf{DGMod}\, A) = \mathbf{D}(\mathsf{Mod}\, A)$.

Derived functors are defined as in the ring case, and there are enough K-injectives, K-projectives and K-flats in $\tilde{\mathbf{K}}(\mathsf{DGMod}\, A)$.

Let $A \to B$ be a homomorphism of DG algebras. There are additive functors

$$B \otimes_A - : \ \mathsf{DGMod}\, A \rightleftarrows \mathsf{DGMod}\, B \ : \mathrm{rest}_{B/A},$$

where $\mathrm{rest}_{B/A}$ is the forgetful functor. These are adjoint. We get induced derived functors

$$B \otimes^{\mathrm{L}}_A - : \ \tilde{\mathbf{D}}(\mathsf{DGMod}\, A) \rightleftarrows \tilde{\mathbf{D}}(\mathsf{DGMod}\, B) \ : \mathrm{rest}_{B/A} \tag{5.2}$$

that are also adjoint.

Proposition 5.3. If $A \to B$ is a quasi-isomorphism, then the functors (5.2) are equivalences.

We say that A is *strongly commutative* if $b \cdot a = (-1)^{i+j} \cdot a \cdot b$ and $c^2 = 0$ for all $a \in A^i$, $b \in A^j$ and $c \in A^k$, where k is odd. We call A *nonpositive* if $A^i = 0$ for al $i > 0$.

Let $f : A \to B$ be a homomorphism between nonpositive strongly commutative DG algebras. A K-flat DG algebra resolution of B relative to A is a factorization of f into $A \xrightarrow{g} \tilde{B} \xrightarrow{h} B$, where h is a quasi-isomorphism, and $\tilde{B}$ is a K-flat DG A-module. Such resolutions exist.

Example 5.4. Take $A = \mathbb{Z}$ and $B := \mathbb{Z}/(6)$. We can take $\tilde{B}$ to be the Koszul complex

$$\tilde{B} := (\cdots 0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to 0 \cdots)$$

concentrated in degrees -1 and 0.

Example 5.5. For a homomorphism of commutative rings $A \to B$, the Hochschild cohomology of B relative to A is the cohomology of the complex

$$\mathrm{RHom}_{\tilde{B} \otimes_A \tilde{B}}(B, B),$$

where $\tilde{B}$ is a K-flat resolution as above.

6. Commutative dualizing complexes

I will talk about dualizing complexes over commutative rings. There is a richer theory for schemes, but there is not enough time for it. See [Hartshorne 1966;

Yekutieli 1992b; Neeman 1996; Yekutieli 2010; Alonso Tarrío et al. 1999; Lipman and Hashimoto 2009], for example.

Let A be a noetherian commutative ring. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the subcategory of $\mathbf{D}(\mathsf{Mod}\, A)$ consisting of bounded complexes whose cohomologies are finitely generated A-modules. This is a full triangulated subcategory.

A complex $M \in \mathbf{D}(\mathsf{Mod}\, A)$ is said to have *finite injective dimension* if it has a bounded injective resolution. Namely there is a quasi-isomorphism $M \to I$ for some bounded complex of injective A-modules I. Note that such I is a K-injective complex.

Take any $M \in \mathbf{D}(\mathsf{Mod}\, A)$. Because A is commutative, we have a triangulated functor

$$\mathrm{RHom}_A(-, M) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A).$$

Compare Example 4.10.

Definition 6.1. A *dualizing complex* over A is a complex $R \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ with finite injective dimension, such that the canonical morphism

$$A \to \mathrm{RHom}_A(R, R)$$

in $\mathbf{D}(\mathsf{Mod}\, A)$ is an isomorphism.

If we choose a bounded injective resolution $R \to I$, then there is an isomorphism of triangulated functors

$$\mathrm{RHom}_A(-, R) \cong \mathrm{Hom}_A(-, I).$$

Example 6.2. Assume A is a *Gorenstein ring*, namely the free module $R := A$ has finite injective dimension. There are plenty of Gorenstein rings; for instance any regular ring is Gorenstein. Then $R \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, and the reflexivity condition holds:

$$\mathrm{RHom}_A(R, R) \cong \mathrm{Hom}_A(A, A) \cong A.$$

We see that the module $R = A$ is a dualizing complex over the ring A.

Here are several important results from [Hartshorne 1966].

Theorem 6.3 (duality). Suppose R is a dualizing complex over A. Then the triangulated functor

$$\mathrm{RHom}_A(-, R) : \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$$

is an equivalence.

Theorem 6.4 (uniqueness). Suppose R and R' are dualizing complexes over A, and $\operatorname{Spec} A$ is connected. Then there is an invertible module P and an integer n such that $R' \cong R \otimes_A P[n]$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Theorem 6.5 (existence). If A has a dualizing complex, and B is a finite type A-algebra, then B has a dualizing complex.

7. Noncommutative dualizing complexes

In the last three sections of the paper we concentrate on noncommutative rings. Before going into the technicalities, here is a brief motivational preface.

Recall that one of the important tools of commutative ring theory is *localization at prime ideals*. For instance, a noetherian local commutative ring A, with maximal ideal $\mathfrak{m}$, is called a *regular local ring* if

$$\dim A = \operatorname{rank}_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

(Here dim is Krull dimension.) A noetherian commutative ring A is called *regular* if all its local rings $A_{\mathfrak{p}}$ are regular local rings.

It is known that regularity can be described in homological terms. Indeed, if $\dim A < \infty$, then it is regular if and only if it has *finite global cohomological dimension*. Namely there is a natural number d, such that $\operatorname{Ext}^i_A(M, N) = 0$ for all $i > d$ and $M, N \in \operatorname{Mod} A$.

Now consider a noetherian noncommutative ring A. (This is short for: A is not-necessarily-commutative, and left-and-right noetherian.) Localization of A is almost never possible (for good reasons). A very useful substitute for localization (and other tools of commutative rings theory) is *noncommutative homological algebra*. By this we mean the study of the derived functors $\operatorname{RHom}_A(-,-)$, $\operatorname{RHom}_{A^{\mathrm{op}}}(-,-)$ and $-\otimes^{\mathrm{L}}_A -$ of formulas (7.2), (7.3) and (8.1) respectively. Here A^{op} is the opposite ring (the same addition, but multiplication is reversed). The homological criterion of regularity from the commutative framework is made the definition of regularity in the noncommutative framework – see Definition 7.1 below. This definition is the point of departure of *noncommutative algebraic geometry* of M. Artin et. al. (see the survey paper [Stafford and van den Bergh 2001]). A surprising amount of structure can be expressed in terms of noncommutative homological algebra. A few examples are sprinkled in the text, and many more are in the references.

Definition 7.1. A noncommutative ring A is called *regular* if there is a natural number d, such that $\operatorname{Ext}^i_A(M, N) = 0$ and $\operatorname{Ext}^i_{A^{\mathrm{op}}}(M', N') = 0$ for all $i > d$, $M, N \in \operatorname{Mod} A$ and $M', N' \in \operatorname{Mod} A^{\mathrm{op}}$.

For the rest of this section A is a noncommutative noetherian ring. For technical reasons we assume that it is an algebra over a field $\mathbb{K}$.

We denote by $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ the enveloping algebra. Thus $\operatorname{Mod} A^{\mathrm{op}}$ is the category of right A-modules, and $\operatorname{Mod} A^{\mathrm{e}}$ is the category of $\mathbb{K}$-central A-bimodules.

Any $M \in \mathsf{Mod}\, A^{\mathrm{e}}$ gives rise to $\mathbb{K}$-linear functors

$$\mathrm{Hom}_A(-, M) : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A^{\mathrm{op}}$$

and

$$\mathrm{Hom}_{A^{\mathrm{op}}}(-, M) : (\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{Mod}\, A.$$

These functors can be right derived, yielding $\mathbb{K}$-linear triangulated functors

$$\mathrm{RHom}_A(-, M) : \mathbf{D}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}}) \tag{7.2}$$

and

$$\mathrm{RHom}_{A^{\mathrm{op}}}(-, M) : \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{D}(\mathsf{Mod}\, A). \tag{7.3}$$

One way to construct these derived functors is to choose a K-injective resolution $M \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$. Then (because A is flat over $\mathbb{K}$) the complex I is K-injective over A and over A^{op}, and we get

$$\mathrm{RHom}_A(-, M) \cong \mathrm{Hom}_A(-, I)$$

and

$$\mathrm{RHom}_{A^{\mathrm{op}}}(-, M) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(-, I).$$

Note that even if A is commutative, this setup is still meaningful – not all A-bimodules are A-central!

Definition 7.4 ([Yekutieli 1992a]). A *noncommutative dualizing complex* over A is a complex $R \in \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ satisfying these three conditions:

(i) The cohomology modules $\mathrm{H}^q(R)$ are finitely generated over A and over A^{op}.

(ii) The complex R has finite injective dimension over A and over A^{op}.

(iii) The canonical morphisms

$$A \to \mathrm{RHom}_A(R, R)$$

and

$$A \to \mathrm{RHom}_{A^{\mathrm{op}}}(R, R)$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ are isomorphisms.

Condition (ii) implies that R has a "bounded bi-injective resolution", namely there is a quasi-isomorphism $R \to I$ in $\mathbf{K}(\mathsf{Mod}\, A^{\mathrm{e}})$, with I a bounded complex of bimodules that are injective on both sides.

Theorem 7.5 (duality [Yekutieli 1992a]). Suppose R is a noncommutative dualizing complex over A. Then the triangulated functor

$$\mathrm{RHom}_A(-, R) : \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$$

is an equivalence, with quasi-inverse $\mathrm{RHom}_{A^{\mathrm{op}}}(-, R)$.

Existence and uniqueness are much more complicated than in the noncommutative case. I will talk about them later.

Example 7.6. The noncommutative ring A is called *Gorenstein* if the bimodule A has finite injective dimension on both sides. It is not hard to see that A is Gorenstein if and only if it has a noncommutative dualizing complex of the form $P[n]$, for some integer n and *invertible bimodule* P. Here invertible bimodule is in the sense of Morita theory, namely there is another bimodule $P^\vee$ such that

$$P \otimes_A P^\vee \cong P^\vee \otimes_A P \cong A$$

in $\mathsf{Mod}\, A^{\mathrm{e}}$. Any regular ring is Gorenstein.

For more results about noncommutative Gorenstein rings see [Jørgensen 1999] and [Jørgensen and Zhang 2000].

8. Tilting complexes and derived Morita theory

Let A and B be noncommutative algebras over a field $\mathbb{K}$. Suppose $M \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ and $N \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. The left derived tensor product

$$M \otimes^{\mathrm{L}}_B N \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}}) \tag{8.1}$$

exists. It can be constructed by choosing a resolution $P \to M$ in $\mathbf{K}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, where P is a complex that's K-projective over B^{op}; or by choosing a resolution $Q \to N$ in $\mathbf{K}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, where Q is a complex that's K-projective over B.

Here is a definition generalizing the notion of invertible bimodule. It is due to Rickard [1989; 1991].

Definition 8.2. A complex

$$T \in \mathbf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

is called a *two-sided tilting complex* over A-B if there exists a complex

$$T^\vee \in \mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$$

such that

$$T \otimes^{\mathrm{L}}_B T^\vee \cong A$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and

$$T^\vee \otimes^{\mathrm{L}}_A T \cong B$$

in $\mathbf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$.

When $B = A$ we say that T is a two-sided tilting complex over A.

The complex $T^\vee$ is called a quasi-inverse of T. It is unique up to isomorphism in $\mathbf{D}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. Indeed we have this result:

Proposition 8.3. Let T be a two-sided tilting complex.

(1) The quasi-inverse $T^\vee$ is isomorphic to $\mathrm{RHom}_A(T, A)$.

(2) T has a bounded bi-projective resolution $P \to T$.

Definition 8.4. The algebras A and B are said to be *derived Morita equivalent* if there is a $\mathbb{K}$-linear triangulated equivalence

$$\mathbf{D}(\mathsf{Mod}\, A) \approx \mathbf{D}(\mathsf{Mod}\, B).$$

Theorem 8.5 [Rickard 1991]. The $\mathbb{K}$-algebras A and B are derived Morita equivalent if and only if there exists a two-sided tilting complex over A-B.

Here is a result relating dualizing complexes and tilting complexes.

Theorem 8.6 (uniqueness [Yekutieli 1999]). Suppose R and R' are noncommutative dualizing complexes over A. Then the complex

$$T := \mathrm{RHom}_A(R, R')$$

is a two-sided tilting complex over A, and

$$R' \cong R \otimes^{\mathrm{L}}_A T$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

It is easy to see that if T_1 and T_2 are two-sided tilting complexes over A, then so is $T_1 \otimes^{\mathrm{L}}_A T_2$. This leads to the next definition.

Definition 8.7 [Yekutieli 1999]. Let A be a noncommutative $\mathbb{K}$-algebra. The *derived Picard group* of A is the group $\mathrm{DPic}_{\mathbb{K}}(A)$ whose elements are the isomorphism classes (in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$) of two-sided tilting complexes. The multiplication is induced by the operation $T_1 \otimes^{\mathrm{L}}_A T_2$, and the identity element is the class of A.

Here is a consequence of Theorem 8.6.

Corollary 8.8. Suppose the noncommutative $\mathbb{K}$-algebra A has at least one dualizing complex. Then operation $R \otimes^{\mathrm{L}}_A T$ induces a simply transitive right action of the group $\mathrm{DPic}_{\mathbb{K}}(A)$ on the set of isomorphism classes of dualizing complexes.

It is natural to ask about the structure of the group $\mathrm{DPic}(A)$.

Theorem 8.9 [Rouquier and Zimmermann 2003; Yekutieli 1999]. If the ring A is either commutative (with nonempty connected spectrum) or local, then

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}.$$

Here $\mathrm{Pic}_{\mathbb{K}}(A)$ is the noncommutative Picard group of A, made up of invertible bimodules. If A is commutative, then

$$\mathrm{Pic}_{\mathbb{K}}(A) \cong \mathrm{Aut}_{\mathbb{K}}(A) \ltimes \mathrm{Pic}_A(A),$$

where $\mathrm{Pic}_A(A)$ is the usual (commutative) Picard group of A. A noncommutative ring A is said to be local if $A/\mathfrak{r}$ is a simple artinian ring, where $\mathfrak{r}$ is the Jacobson radical.

For nonlocal noncommutative rings the group $\mathrm{DPic}_{\mathbb{K}}(A)$ is bigger. See the paper [Miyachi and Yekutieli 2001] for some calculations. These calculations are related to CY-dimensions of some rings; see Example 9.7.

9. Rigid dualizing complexes

The material in this final section is largely due to Van den Bergh [van den Bergh 1997]. His results were extended by J. Zhang and myself. Again A is a noetherian noncommutative algebra over a field $\mathbb{K}$, and $A^{\mathrm{e}} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$.

Take $M \in \mathsf{Mod}\, A^{\mathrm{e}}$. Then the $\mathbb{K}$-module $M \otimes_{\mathbb{K}} M$ has four commuting actions by A, which we arrange as follows. The algebra $A^{\mathrm{e;\,in}} := A^{\mathrm{e}}$ acts on $M \otimes_{\mathbb{K}} M$ by

$$(a_1 \otimes a_2) \cdot_{\mathrm{in}} (m_1 \otimes m_2) := (m_1 \cdot a_2) \otimes (a_1 \cdot m_2),$$

and the algebra $A^{\mathrm{e;\,out}} := A^{\mathrm{e}}$ acts by

$$(a_1 \otimes a_2) \cdot_{\mathrm{out}} (m_1 \otimes m_2) := (a_1 \cdot m_1) \otimes (m_2 \cdot a_2).$$

The bimodule A is viewed as an object of $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ in the obvious way.

Now take $M \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. We define the *square* of M to be the complex

$$\mathrm{Sq}_{A/\mathbb{K}}(M) := \mathrm{RHom}_{A^{\mathrm{e;\,out}}}(A, M \otimes_{\mathbb{K}} M) \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e;\,in}}).$$

We get a functor

$$\mathrm{Sq}_{A/\mathbb{K}} : \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}}) \to \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}}).$$

This is not an additive functor. Indeed, it is a quadratic functor: given an element $a \in \mathrm{Z}(A)$ and a morphism $\phi : M \to N$ in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, one has

$$\mathrm{Sq}_{A/\mathbb{K}}(a \cdot \phi) = \mathrm{Sq}_{A/\mathbb{K}}(\phi \cdot a) = a^2 \cdot \mathrm{Sq}_{A/\mathbb{K}}(\phi).$$

Note that the cohomologies of $\mathrm{Sq}_{A/\mathbb{K}}(M)$ are

$$\mathrm{H}^j(\mathrm{Sq}_{A/\mathbb{K}}(M)) = \mathrm{Ext}^j_{A^{\mathrm{e}}}(A, M \otimes_{\mathbb{K}} M),$$

so they are precisely the Hochschild cohomologies of $M \otimes_{\mathbb{K}} M$.

A *rigid complex* over A (relative to $\mathbb{K}$) is a pair (M, ρ) consisting of a complex $M \in \mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, and an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Let (M, ρ) and (N, σ) be rigid complexes over A. A *rigid morphism*

$$\phi : (M, \rho) \to (N, \sigma)$$

is a morphism $\phi : M \to N$ in $\mathbf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, such that the diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\phi"'] & \mathrm{Sq}_{A/\mathbb{K}}(M) \arrow[d, "\mathrm{Sq}_{A/\mathbb{K}}(\phi)"] \\ N \arrow[r, "\sigma"] & \mathrm{Sq}_{A/\mathbb{K}}(N) \end{tikzcd}$$

is commutative.

Definition 9.1 ([van den Bergh 1997]). A *rigid dualizing complex* over A (relative to $\mathbb{K}$) is a rigid complex (R, ρ) such that R is a dualizing complex.

Theorem 9.2 (uniqueness [van den Bergh 1997; Yekutieli 1999]). Suppose (R, ρ) and (R', ρ') are both rigid dualizing complexes over A. Then there is a unique rigid isomorphism

$$\phi : (R, \rho) \xrightarrow{\simeq} (R', \rho').$$

As for existence, let me first give an easy case.

Proposition 9.3. If A is finite over its center $\mathrm{Z}(A)$, and $\mathrm{Z}(A)$ is finitely generated as $\mathbb{K}$-algebra, then A has a rigid dualizing complex.

Actually, in this case it is quite easy to write down a formula for the rigid dualizing complex.

In the next existence result, by a filtration $F = \{F_i(A)\}_{i \in \mathbb{N}}$ of the algebra A we mean an ascending exhaustive filtration by $\mathbb{K}$-submodules, such that $1 \in F_0(A)$ and $F_i(A) \cdot F_j(A) \subset F_{i+j}(A)$. Such a filtration gives rise to a graded $\mathbb{K}$-algebra

$$\mathrm{gr}^F(A) = \bigoplus_{i \geq 0} \mathrm{gr}^F_i(A).$$

Theorem 9.4 (existence, [van den Bergh 1997; Yekutieli and Zhang 2005]). Suppose A admits a filtration F, such that $\mathrm{gr}^F(A)$ is finite over its center $\mathrm{Z}(\mathrm{gr}^F(A))$, and $\mathrm{Z}(\mathrm{gr}^F(A))$ is finitely generated as $\mathbb{K}$-algebra. Then A has a rigid dualizing complex.

This theorem applies to the ring of differential operators $\mathcal{D}(C)$, where C is a smooth commutative $\mathbb{K}$-algebra (and $\operatorname{char}\mathbb{K} = 0$). It also applies to any quotient of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$.

I will finish with some examples.

Example 9.5. Let A be a noetherian $\mathbb{K}$-algebra satisfying these two conditions:

- A is smooth, namely the A^{e}-module A has finite projective dimension.
- There is an integer n such that

$$\operatorname{Ext}^j_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} A & \text{if } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then A is a regular ring (Definition 7.1), and the complex $R := A[n]$ is a rigid dualizing complex over A. Such an algebra A is called an *n-dimensional Artin–Schelter regular algebra*, or an *n-dimensional Calabi–Yau algebra*.

Example 9.6. Let $\mathfrak{g}$ be an n-dimensional Lie algebra, and $A := \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra. Then the rigid dualizing complex of A is $R := A^{\sigma}[n]$, where A^{σ} is the trivial bimodule A, twisted on the right by an automorphism σ. Using the Hopf structure of A we can express A^{σ} like this:

$$A^{\sigma} \cong \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \textstyle\bigwedge^n(\mathfrak{g}),$$

the twist by the 1-dimensional representation $\bigwedge^n(\mathfrak{g})$. See [Yekutieli 2000]. So A is a *twisted Calabi–Yau* algebra. If $\mathfrak{g}$ is semi-simple then there is no twist, and A is Calabi–Yau. This was used by Van den Bergh in his duality for Hochschild (co)homology [van den Bergh 1998].

Example 9.7. Let

$$A := \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$$

the 2×2 upper triangular matrix algebra. The rigid dualizing complex here is

$$R := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K}).$$

It is known that

$$R \otimes^{\mathrm{L}}_A R \otimes^{\mathrm{L}}_A R \cong A[1]$$

in $\mathsf{D}(\operatorname{Mod} A^{\mathrm{e}})$. So A is a *Calabi–Yau algebra of dimension* $\frac{1}{3}$. See [Yekutieli 1999; Miyachi and Yekutieli 2001].

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