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Preface

Inverse problems are those where one tries to obtain information about a system (such as a patient's body, an industrial object, or the Earth's interior) without taking it apart: the system is a "black box" whose properties we would like to reconstruct from its response to appropriate stimuli, such as waves or fields of various types.

Such problems lie at the heart of contemporary scientific inquiry and technological development. Their study has led to the development of a vast array of techniques in medical, geophysical, and industrial imaging, radar, sonar, and other areas. Practical applications of these techniques include the early detection of cancer and pulmonary edema, location of oil and mineral deposits in the Earth's interior, creation of astrophysical images from telescope data, detection of cracks and interfaces within materials, shape optimization, model identification in growth processes, and much more.

Most inverse problems arise from a physical situation modeled by a partial differential equation. The inverse problem is to determine parameters of the equation given some information about the solutions. The analysis of such problems brings together diverse areas of mathematics such as complex analysis, differential geometry, harmonic analysis, integral geometry, microlocal analysis, numerical analysis, optimization, partial differential equations, and probability. It is a fertile area for interaction between pure and applied mathematics. This book includes several chapters on some of the topics discussed at the Inverse Problems and Applications Semester held at MSRI in Fall 2010.

A prototypical example of an inverse boundary problem for an elliptic equation is the by now classical Calderón problem, also called electrical impedance tomography (EIT). Calderón proposed the problem in the mathematical literature in 1980. In EIT one attempts to determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. The information is encoded in the Dirichlet-to-Neumann map associated to the conductivity equation. EIT arises in several applications including geophysical prospection (Calderón's original motivation) and medical imaging. In the last 30 years or so there has been remarkable progress on this problem. In the last few years this includes the two-dimensional problem, the case of partial data, and the discrete problem. The chapter by Astala, Lassas and Päivärinta describes some of this progress in the two-dimensional case. The chapter by Borcea, Druskin, Guevara Vasquez and Mamonov discusses recent progress on discrete network problems and their approximation to the continuous problem, also in the two-dimensional case. The chapter by Guillarmou and Tzou deals with the Calderón problem on two-dimensional manifolds and, more generally, with inverse boundary value problems for the Schrödinger equation on Riemann surfaces, including the case of partial data. The chapter by Salo gives a detailed account of progress achieved for the anisotropic problem in dimensions three and higher. The chapter by Wang and Zhou considers the determination of the location of discontinuous electromagnetic or elastic parameters using the enclosure method.

New inverse problems and methodologies arise all the time because of the applications. For example, in medical imaging, there has been considerable interest in recent years in multiwave methods, also called hybrid methods, which combine a high-resolution modality with a high-contrast one through a physical principle. For example, in breast imaging ultrasound provides high (submillimeter) resolution, but suffers from low contrast. On the other hand, many tumors absorb much more energy from electromagnetic waves than healthy cells. Thus using electromagnetic waves offers very high contrast. Examples of novel hybrid imaging methods are photoacoustic tomography (PAT), thermoacoustic tomography (TAT), ultrasound modulation tomography, transient elastography, and magnetic resonance elastography. We describe briefly PAT and TAT. PAT consists of exposing tissues to relatively harmless optical radiation that causes temperature increases in the millikelvin range, resulting in the generation of propagating ultrasound waves (the photoacoustic effect). Such ultrasound waves are readily measurable. The inverse problem then consists of reconstructing the optical properties of the tissue. In TAT, low-frequency microwaves, with wavelengths on the order of 1 m, are sent into the medium. The rationale for using the latter frequencies is that they are less absorbed than optical frequencies. PAT and TAT offer potential breakthroughs in the clinical application of hybrid imaging modalities to early detection of cancer, functional imaging, and molecular imaging. The inverse problem for these two modalities has two steps. The first step is to solve a well-posed inverse boundary problem for the wave equation. The chapter by Stefanov and Uhlmann surveys recent progress for this step, including the case of discontinuous sound speeds. From the first step one obtains different internal functionals for the elastic, electromagnetic, or optical properties of the medium, depending on the imaging modality. The inverse problem is to recover the parameters of the medium from knowledge of these internal functionals. The chapter by Bal summarizes in detail what is known about this second step.

An outstanding inverse problem in geophysics consists in determining the inner structure of the Earth from measurements of travel times of seismic waves.

From a mathematical point of view, the inner structure of the Earth is modelled by a Riemannian metric, and travel times by the lengths of geodesics between boundary points. This gives rise to a typical geometric inverse problem: is it possible to determine a Riemannian metric from its boundary distance function? The geodesic ray transform, where one integrates a function or a tensor field along geodesics of a Riemannian metric, is closely related to the boundary rigidity problem. The integration of a function along geodesics is the linearization of the boundary rigidity problem in a fixed conformal class. The standard X-ray transform, where one integrates a function along straight lines, corresponds to the case of the Euclidean metric and is the basis of medical imaging techniques such as CT and PET. The case of integration along more general geodesics arises in geophysical imaging in determining the inner structure of the Earth, since the speed of elastic waves generally increases with depth, thus curving the rays back to the surface. It also arises in ultrasound imaging, where the Riemannian metric models the anisotropic index of refraction. In tensor tomography problems one would like to determine a symmetric tensor field up to natural obstruction from its integrals over geodesics. Recently there has been much activity in the study of these transforms. Besides their importance in imaging technology, they arise naturally in various inverse problems in geometry as explained above.

In the case of Euclidean space with the Euclidean metric the attenuated ray transform is the basis of the medical imaging technology of SPECT and has been extensively studied. There are two natural directions in which this transform can be extended: one is to replace Euclidean space by a Riemannian manifold, and the other is to consider the case of systems where the attenuation is given, for example, by a unitary connection. The chapter by Paternain on inverse problems for connections deals with this latter case as well as other geometric inverse problems for connections.

One of the fascinating aspects of inverse problems is the continuous interplay between pure and applied mathematics. This interplay has been particularly noticeable in the applications of microlocal analysis (MA) to inverse problems. MA — which is, roughly speaking, local analysis in phase space — was developed about 40 years ago by Hörmander, Maslov, Sato, and many others in order to understand the propagation of singularities of solutions of partial differential equations. The early roots of MA were in the theory of geometrical optics. Microlocal analysis has been used successfully in determining the singularities of medium parameters in several inverse problems ranging from X-ray tomography to reflection seismology and electrical impedance tomography.

Brytik, De Hoop, and Van der Hilst's chapter considers applications of MA to geophysics. In particular it contains a new analysis of elastic reverse time migration, using MA as a tool, with applications to exploration seismology and global seismology. Another area where MA has been applied successfully in recent years is inverse spectral problems centered around Marc Kac's problem: can one hear the shape of a drum? The chapter by Datchev and Hezari surveys recent progress on this and other inverse spectral questions as well as inverse problems for resonances.

Vasy writes in his chapter about a compelling new way he found of doing scattering theory for the Laplacian on conformally compact spaces, including nontrapping high-energy bounds for the analytic continuation of the resolvent in appropriate circumstances, by appropriately extending the problem to a larger space. The resulting problems are not elliptic, but fit into a very nice microlocal framework on manifolds without boundary. The microlocal machinery being used is also very useful in other settings, including asymptotically hyperbolic spaces and Lorentzian geometries including Kerr–de Sitter spaces. Vasy develops this approach in detail in his chapter for the case of asymptotically hyperbolic spaces.

In scattering theory there has been a flurry of activity recently about transmission eigenvalues — wavenumbers at which the scattering operator has a nontrivial kernel and cokernel. The chapter by Cakoni and Haddar describes the recent developments as well as several open problems.

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