Theory of the small

Yellow-Brown Hackenbush

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ABSTRACT. This game is played on a sum of strings. In its "restricted" form, Left, at her turn, picks a bichromatic string and removes its highest ye**LL**ow branch. Right, at his turn, picks a bichromatic string and removes its highest b**R**own branch. As in the well-known game of b**L**ue-**R**ed Hackenbush, all higher branches, being disconnected, also disappear. But in yellow-brown Hackenbush, unlike blue-red Hackenbush, all moves on monochromatic strings are *illegal*. This makes all values of yellow-brown Hackenbush all-small.

This paper presents an explicit solution of restricted yellow-brown Hackenbush. The values are sums of basic infinitesimals that have appeared in many other games found in *Winning Ways*.

Yellow-Brown (YB) Hackenbush is played on sums of strings. Each mixed string is played analogously to LR Hackenbush. Left can remove a ye**LL**ow branch, and all other branches (if any) above it; Right can remove a b**R**own branch and all other branches (if any) above it. But yellow-brown strings differ from blue-red strings in an important respect:

Neither player is allowed to move on any monochromatic string.

That rule ensures that all stopping positions are 0, and that all YB values are infinitesimal.

There are (at least) two variations of YB Hackenbush: restricted and unrestricted. In the restricted variation, each player is allowed at most a single option on any string, namely, his branch which is highest above the ground. Although this restriction would have no effect on the values of LR Hackenbush, it has a major effect on YB Hackenbush. In the unrestricted YB Hackenbush, either player can move to 0 by playing his lowest branch, so all nonzero values are confused with 0.

We now present a complete solution for *restricted YB strings*.

To each YB string we may associate a number x which is the value of the corresponding LR Hackenbush string. It is not hard to see, recursively, that the value of the YB string is

$$v = \int_0^0 x$$

Although this operator, $\int_0^0$, is nonlinear, it can be used to compute the value v. This value can be expressed in terms of a basis of standard infinitesimals. The key to accomplishing this is another number, namely

$$y = \lceil x \rceil + \tfrac{1}{2} - x.$$

Here are the values corresponding to some short YB strings:

String	x	y	v
Y	1	.1	0
YYBBB	1.001	.011	$\Uparrow *$
YYBB	1.01	.01	$\uparrow$
YYBBYB	1.0101	.0011	$2.\uparrow[2]$
YYBBY	1.011	.001	$\uparrow[2]$
YYBBYY	1.0111	.0001	$\uparrow[3]$
YYB	1.1	0	$*$

where the basis infinitesimals in the v column are defined by

$$\begin{aligned}
\uparrow[1] &= 0 \mid * \\
\uparrow[2] &= \uparrow[1] \mid * \\
\uparrow[3] &= \uparrow[2] \mid * \\
&\cdots \\
\uparrow[n+1] &= \uparrow[n] \mid *
\end{aligned}$$

In general, if $y = \sum_{i=1}^{n} Y_i 2^{-i}$, where each Y_i equals 0 or 1, with $Y_1 = 0$ and $Y_n = 1$, then our asserted value is

$$v(y) = * + \sum_{i=l}^{n} Y_i \left(\uparrow \left[\sum_{j=1}^{i} (1 - Y_i) \right] * \right),$$

where $\uparrow[n]$ is defined above. So, for example, consider the string

$$\mathrm{Y\ Y\ B\ B\ B\ Y\ B\ Y\ Y\ B\ Y\ B\ B\ B\ Y}$$

Using the well-known rule for Blue-Red Hackenbush strings (see *Winning Ways*, vol. 1), we have the binary expansion of x, namely

$$\begin{array}{rccccccccccccccc}
 & \mathrm{Y} & \mathrm{Y} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{Y} & \mathrm{B} & \mathrm{Y} & \mathrm{Y} & \mathrm{B} & \mathrm{Y} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{Y} \\
x = & 1 & \multicolumn{2}{c}{\underbrace{\qquad}_{\cdot}} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1\ 1
\end{array}$$

from which we proceed as follows:

$$\begin{array}{rll}
x = & 1 \quad \cdot & 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1 \\
y = & \quad\ \ \cdot & 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \underbrace{1\ 1\ 1}\ 0\ 1 \\
v = & * & +\ \uparrow[1]* \\
 & & \quad +\ \uparrow[2]* \\
 & & \qquad +\ \uparrow[4]* \\
 & & \qquad\quad +\ 3.\uparrow[5]* \\
 & & \qquad\qquad +\ \uparrow[6]*
\end{array}$$

SKETCHED PROOF. The negatives of the Left incentives of the basis infinitesimals are positive infinitesimals of increasingly higher orders:

$$\uparrow\ =\ -\Delta^{\mathrm{L}}\uparrow[1]\ \gg\ -\Delta^{\mathrm{L}}\uparrow[2]\ \gg\ \cdots\ \gg\ \Delta^{\mathrm{L}}\uparrow[j]\ >\ 0$$

In real analysis, one expects equations such as

$$\varepsilon \gg \varepsilon^2 \gg \cdots \gg \varepsilon^k > 0,$$

so, by analogy, it is common to denote $-\Delta^{\mathrm{L}}\uparrow[j]$ by $\uparrow^j$, even though no multiplication is defined nor intended.

Then

$$\begin{aligned}\uparrow[j] &= \uparrow[j-1] + \uparrow^j \\ &= \sum_{i=1}^{j} \uparrow^i .\end{aligned}$$

For every j,

$$* \not\gtrless \sum_{i=1}^{j} \uparrow^i$$

but

$$*\ <\ \sum_{i=1}^{j-1} \uparrow^i + \uparrow^j + \uparrow^j .$$

Both sequences of incentives increase

$$\Delta^{\mathrm{L}}(\uparrow[j]) < \Delta^{\mathrm{L}}(\uparrow[j+1])$$
$$\Delta^{\mathrm{R}}(\uparrow[j]) < \Delta^{\mathrm{R}}(\uparrow[j+1]).$$

Therefore, in any sum of nonnegative integer multiples of $\uparrow[j]$, each player's dominant move is on a term with maximum superscript.

The cases

$$y = .1, \qquad v = \uparrow[0] = 0$$
$$\text{and} \quad y = 0, \qquad v = *$$

are degenerate. In the general nondegenerate case, $0 < y < \frac{1}{2}$,

$$y = \sum_{i=1}^{n} Y_i 2^{-i} \quad \text{for each } Y_i = 0 \text{ or } 1.$$

Let m be the integer for which $Y_m = 0$, but $Y_i = 1$ for $m < i \le n$.

Let

$$k = \sum_{i=1}^{n}(1 - Y_i) = \sum_{i=1}^{m}(1 - Y_i).$$

Then

$$v(y) = * + \sum_{i=1}^{m} Y_i \left(\uparrow \left[\sum_{j=1}^{i} (1 - Y_j) \right] * \right) + (n-m).(\uparrow[k]*)$$

We now explore properties of this asserted value, leading to a sketched proof that it is indeed the value of the corresponding restricted YB Hackenbush string.

We notice that since $0 < y < \frac{1}{2}$, $v(y)$ is positive. From $v(y)$, Right has a unique dominant incentive, $\uparrow[k]*$. Left has two dominant incentives, $*$ and $\downarrow^{k}$. However, Left's move of incentive $*$ reverses to another position whose incentive is dominated by $\downarrow^{k}$, and so we have the equation

$$v(y) = \left\{ v(y) + \downarrow^{k} \;\middle|\; v(y) - \uparrow[k]* \right\}. \tag{0-1}$$

If $n = m + 1$, this is canonical form. However, if $n \ge m + 2$, then Left's move is reversible. It continues to reverse to 0, and the canonical form is

$$v(y) = \left\{ 0 \;\middle|\; v(y) - \uparrow[k]* \right\}, \quad \text{if } n \ge m + 2. \tag{0-2}$$

The canonical positions of the number y include:

$$y^{\mathrm{L}} = y - 2^{-n},$$
$$y^{\mathrm{R}} = y + 2^{-n},$$
$$y^{\mathrm{RL}} = y + 2^{-n} - 2^{-m},$$
$$y^{\mathrm{L}} = y^{\mathrm{RL}} + (2^{-m} - 2^{1-n}).$$

The asserted YB values satisfy

$$\begin{aligned} v(y) &= v(y^{\mathrm{L}}) + {\uparrow}[k]*, \\ v(y^{\mathrm{R}}) &= v(y^{\mathrm{RL}}) + {\uparrow}[k-1]*, \\ v(y) &= v(y^{\mathrm{RL}}) = (n-m).({\uparrow}[k]*), \\ v(y^{\mathrm{L}}) &= v(y^{\mathrm{RL}}) + (n-m-1).({\uparrow}[k]*), \end{aligned} \tag{0-3}$$

and

$$v(y) > v(y^{\mathrm{R}}). \tag{0-4}$$

If $n = m+1$, then in view of (0-3), (0-1) becomes

$$v(y) = \{v(y^{\mathrm{R}}) \mid v(y^{\mathrm{L}})\} \tag{0-5}$$

We next show that (0-5) remains valid if $n \geq m+2$.

From (0-2), we have

$$v(y) = \{0 \mid v(y^{\mathrm{L}})\}.$$

Relation (0-4) implies that $v(y^{\mathrm{R}}) \ngtr v(y)$, and so the Gift Horse principle ensures that

$$\begin{aligned} v(y) &= \{0 \mid v(y^{\mathrm{L}})\} = \{0, v(y^{\mathrm{R}}) \mid v(y^{\mathrm{L}})\} \\ &= \{v(y^{\mathrm{R}}) \mid v(y^{\mathrm{L}})\}, \quad \text{because} \;\; v(y^{\mathrm{R}}) \geq 0. \end{aligned}$$

Thus, (0-5) is valid for any $n > m$. Since $y = \frac{1}{2} - x$, this recursion implies that if there is any value of y for which the asserted YB string value is incorrect, it must be degenerate. But we have verified the degenerate cases, and so the asserted values are correct for all restricted YB strings. □

Historical note

This game arose in the early 1990s when I was studying a variety of overheating operators and trying (in vain) to get more understanding of the conditions under which they are linear and/or monotonic. In my graduate seminars on game theory, we considered numbers overheated from 1 to infinity. We called them "vaporized numbers" and observed their close relationship to numbers overheated from 0 to 0. We then invented this game. Several graduate students participated in those discussions. In spring 1994, I wrote a preliminary version of this paper and distributed it to the class. Shortly thereafter Kuo-Yuen Kao studied some similar games, and included them in his unpublished doctoral thesis at UNC Charlotte.

The invention of Clobber in 2002 rekindled widespread interest in properties of infinitesimals and led me to begin a slow and sporadic search which culminated in finding this paper buried deep in my files.

References

"Sums of Hot and Tepid Combinatorial Games", by Kuo-Yuan Kao, Ph.D. Thesis, University of North Carolina at Charlotte, 1997.

Winning Ways for Your Mathematical Plays, 2nd edition (4 volumes), by Elwyn Berlekamp, John Conway and Richard Guy, A K Peters, Wellesley, MA, 2001–2004.

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