Games of No Chance 3
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# On day $n$

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ABSTRACT. We survey the work done to date about games born by day $n$.

## 1. Introduction

Both number theory and combinatorial game theory are interesting in large part because of the wonderful interplay between algebraic and combinatorial structure. Here we survey some general results that investigate either the additive structure or the partial order of the games born by day $n$.

The games born by day $n$, $\mathcal{G}_n$, have game trees of height at most $n$. More formally, $\mathcal{G}_n$ is defined inductively:

$$\begin{aligned} \mathcal{G}_0 &\stackrel{\text{def}}{=} \{0\}, \\ \mathcal{G}_n &\stackrel{\text{def}}{=} \big\{\{G^L \,|\, G^R\} :\ G^L, G^R \subseteq \mathcal{G}_{n-1}\big\}. \end{aligned}$$

$\mathcal{G}_1$ consists of games whose left and right options are subsets of $\mathcal{G}_0$, i.e., either $\{\}$ or $\{0\}$. This yields four games born by day 1, those being

$$0 = \{\ |\ \}, \quad 1 = \{0\,|\ \}, \quad -1 = \{\ |\,0\}, \quad * = \{0\,|\,0\}.$$

We can draw the partial order of these four games to get

$$\begin{array}{ccccc} & & 1 & & \\ & / & & \backslash & \\ 0 & & & & * \\ & \backslash & & / & \\ & & -1 & & \end{array}$$

On day 2, left and right options are subsets of the day 1 lattice. Since there are 16 subsets of $\mathcal{G}_1$, this yields at most $16 \cdot 16 = 256$ games born by day 2.

<table>
<tr><th></th><th></th><th colspan="6">Right</th></tr>
<tr><th></th><th></th><th>$-1$</th><th>$0, *$</th><th>$0$</th><th>$*$</th><th>$1$</th><th>$\varnothing$</th></tr>
<tr><td rowspan="6">Left</td><td>$1$</td><td>$\pm 1$</td><td>$1\|0, *$</td><td>$1\|0$</td><td>$1\|*$</td><td>$1*$</td><td>$2$</td></tr>
<tr><td>$0, *$</td><td>$0,*\|-1$</td><td>$*2$</td><td>$\uparrow *$</td><td rowspan="2">$\uparrow$</td><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">$1$</td></tr>
<tr><td>$0$</td><td>$0\|-1$</td><td>$\downarrow *$</td><td>$*$</td></tr>
<tr><td>$*$</td><td>$*\|-1$</td><td colspan="2">$\downarrow$</td><td colspan="3" rowspan="3">$0$</td></tr>
<tr><td>$-1$</td><td>$-1*$</td><td colspan="2">$-\frac{1}{2}$</td></tr>
<tr><td>$\varnothing$</td><td>$-2$</td><td colspan="2">$-1$</td></tr>
</table>

**Figure 1.** The 22 games born by day 2 organized by Left and Right options.

Note, however, that we can restrict our attention to only those subsets without dominated options, i.e., the antichains in $\mathcal{G}_1$. There are six such antichains

$$\{1\}, \{0, *\}, \{0\}, \{*\}, \{-1\}, \{\}$$

roughly sorted so that those Left most wishes to be her option list are listed first. This leaves us with at most 36 games born by day 2. Of these 36, many are equal, leaving the 22 distinct games shown in Figure 1.

## 2. Games as a group

Under game addition, although the games born by day $n > 0$ do not form a group, it is natural to investigate the group generated by the games born by day $n$, which we will denote $J_n$. On day 0, we have just the singleton $J_0 = \{0\}$. $\mathcal{G}_1 = \{0, 1, -1, *\}$, and sums of these games consist of integers $n$ and $n*$. Since $* + * = 0$, we have that $J_1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$.

Moews [1991] investigates $J_2$ and $J_3$. He shows that $J_2$ has the basis

$$1/2, *2, A, \uparrow, \alpha, \pm\tfrac{1}{2}, \pm 1,$$

where

$$A = \{1\,|\,0\} - \{1\,|\,*\}, \quad \alpha = \{1\,|\,0\} - \{1\,|\,0, *\}.$$

$A$ has order 4 since $A + A = *$, while $\alpha > 0$ has atomic weight 0 and is therefore linearly independent with $\uparrow$. So, we have that $J_2$ is isomorphic to $\mathbb{Z}^3 \times \mathbb{Z}_4 \times \mathbb{Z}_2^3$.

Let $I_n$ be the group of infinitesimal games within $J_n$. Then $I_2$ is $\mathbb{Z}^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and $J_2/I_2$ is $\mathbb{Z} \times \mathbb{Z}_2^2$.

Moews employs a combination of computation and mathematical ingenuity to describe $J_3/I_3$, but leaves open $I_3$ (and therefore $J_3$.) His key result that

$$J_3/I_3 = \mathbb{Z}^7 \times \mathbb{Z}_4 \times \mathbb{Z}_2^8$$

has a reasonably technical proof.

## 3. Games as a partial order

Games born by day $n$ form a distributive lattice, but the collection of all short games, $\mathcal{G} = \bigcup_{n\geq 0} \mathcal{G}[n]$, is not a lattice [Calistrate et al. 2002]. The key to identifying the lattice structure is to explicitly construct the *join* (or least upper bound) and *meet* (or greatest lower bound) of two elements. Since the partial order is self-dual (i.e., each game has a negative and $G \geq H$ exactly when $-H \geq -G$), we will only state theorems in terms of the join operation, and leave it to the reader to construct the symmetric assertions concerning the meet operation.

For the day $n$ lattice, define the join in terms of the operation

$$\lceil G \rceil \stackrel{\text{def}}{=} \{H \in \mathcal{G}_{n-1} : \ H \not\leq G\}$$

The notation $\lceil G \rceil$, and $G_1 \vee G_2$ below, take the current day $n$ for granted. Then the join of two games is given by

$$G_1 \vee G_2 \stackrel{\text{def}}{=} \left\{G_1^L, G_2^L \mid \lceil G_1 \rceil \cap \lceil G_2 \rceil\right\}$$

Note that $G_1 \vee G_2$ is in $\mathcal{G}_n$ since its left and right options are all in $\mathcal{G}_{n-1}$.

It is now a reasonable graduate level exercise to prove that the join operation above exactly reflects the partial order of games born by day $n$, and that join distributes over a symmetrically defined meet.

The *Hasse diagram* of the lattices for days 1 and 2 is shown on the left side of Figure 2. One property of distributive lattices is that they are *graded* or *ranked*, where the partial order can be drawn with edges only going between adjacent levels. The lattices for days $n \leq 3$ all share the property that the middle level is the widest (i.e., has the most games). It is still open, but should be computationally feasible, to organize and describe the exact structure of the day 3 lattice of 1474 games.

In a lattice, the *join irreducible* elements are those elements that cannot be formed by the join of other elements. Looking at the Hasse diagram of the lattice, a join irreducible element has exactly one element immediately below it in the lattice. (The single element at the bottom is not considered a join irreducible for it is the join of the empty set.) The right side of Figure 2 shows the partial order of the day 2 join irreducibles.

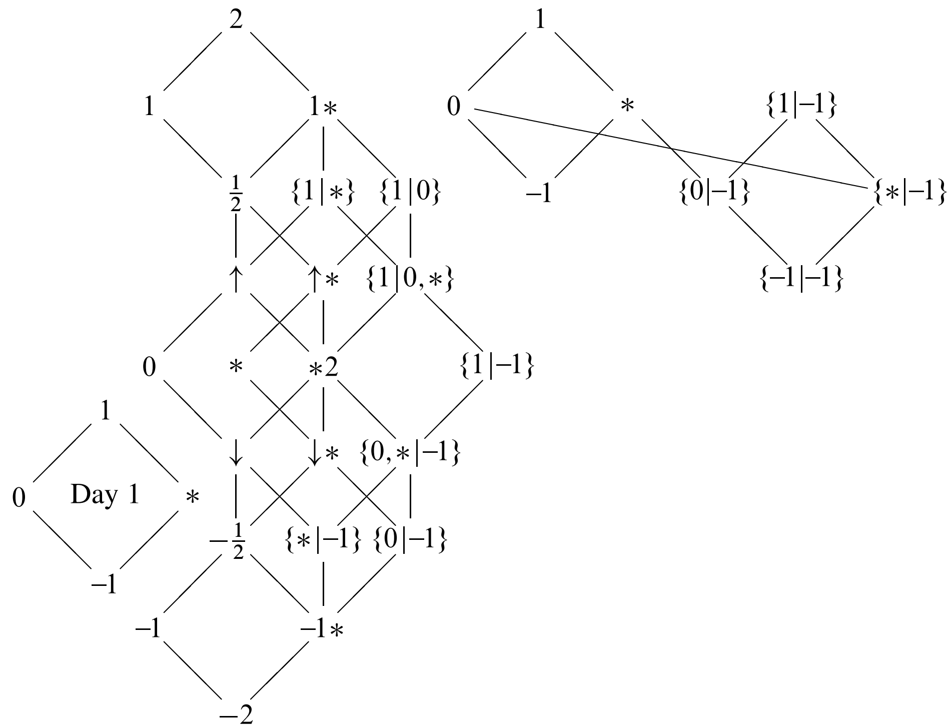


**Figure 2.** Left: day 1 and day 2 lattices. Right: join irreducibles from day 2.

Birkhoff [1940] showed (amazingly) that there is a natural one-to-one correspondence of finite partial orders with finite distributive lattices, where that correspondence is via the partial order on join irreducibles. As shown in [Fraser et al. 2005], the join irreducibles from the $\mathcal{G}_{n+1}$ lattice are exactly those games of the form $g$ or $\{g\,|\,{-n}\}$ where $g \in \mathcal{G}_n$.

As an immediate corollary of this fact (and Birkhoff's construction of the distributive lattice from its join irreducibles), all maximal chains on day $n$ are of length exactly one plus double the number of games born by day $n-1$.

Aaron Siegel [2005] showed that the distributive lattice for $\mathcal{G}_n$ has exactly two automorphism, i.e., one order-preserving symmetry. In particular, he defines a companion $g^\bullet$ of each element $g \in \mathcal{G}_n$ by

$$g^\bullet = \begin{cases} * & \text{if } G = 0,\\ \{0, (G^L)^\bullet \,|\, (G^R)^\bullet\} & \text{if } G > 0,\\ \{(G^L)^\bullet \,|\, (G^R)^\bullet\} & \text{if } G \text{ is incomparable with } 0,\\ \{(G^L)^\bullet \,|\, 0, (G^R)^\bullet\} & \text{if } G < 0, \end{cases}$$

This is the only nontrivial automorphism which preserves the partial order on $\mathcal{G}_n$. Further, this automorphism also preserves birthday (for games other than

0 and $*$) and atomic weight of all-small games. He defines the *longitude* of a game $G$ by the difference in ranks between $G$ and $G \vee G^{\bullet}$; this is some measure of how far $G$ is from the "spine" of self-companions.

## 4. The all-small lattice

An all-small game is one in which Left has an option if and only if Right has one as well. On day 1, 0 and $*$ are all-small, while 1 and $-1$ are not. Day 2 has 7 all-small games:

$$
\begin{tikzcd}
 & \uparrow \arrow[dl, dash] \arrow[dr, dash] & & \uparrow * \arrow[dl, dash] \arrow[dr, dash] & \\
0 & & *2 & & * \\
 & \downarrow \arrow[ul, dash] \arrow[ur, dash] & & \downarrow * \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
$$

In his thesis, Aaron Siegel [2005] proved that, subject to a minor caveat, the all-small games born on day $n$ also form a distributive lattice. The caveat is that one must adjoin a single element to the top (and, symmetrically, bottom) of the lattice which is the join of the two maximal elements $(n-1)\cdot\uparrow$ and $(n-1)\cdot\uparrow *$. This lattice also has the unique nontrivial automorphism given by $g^{\bullet}$ above. There are 67 all-smalls born by day 3, and a figure of the lattice appears in Siegel's thesis. He also computes the 534,483 all-smalls born on day 4 and has found that while the middle level of this lattice remains the largest, its thickest level, as measured by maximum longitude, is not the middle level.

## 5. Counting games

The fact that there are 1474 games born by day 3 has been known for some time. Dean Hickerson found them by hand sometime around 1974, though he may not have been the first. The best known upper and lower bounds on the number of games born by day $n$ for larger values of $n$ are given in [Wolfe and Fraser 2004], and depend upon observations made (in personal communications) by Dean Hickerson and Dan Hoey.

Consider the lattice of games born by $\mathcal{G}_n$. Call a pair $(\mathcal{T}, \mathcal{B})$ of antichains in this lattice *admissible* if $\mathcal{T} > \mathcal{B}$ (i.e., each game in $\mathcal{T}$ exceeds each game in $\mathcal{B}$.) The new games born by day $n+1$ are in one-to-one correspondence with admissible pairs from day $n$. This fact can be used to bound the number of games $g(n)$ born by day $n$ recursively by,

$$g(n+1) \le g(n) + 2^{1+g(n)}.$$

The bound can be tightened somewhat to

$$g(n+1) \le g(n) + 2^{g(n)} + 2,$$

or even further to

$$g(n+1) \le g(n) + \bigl(g(n-1)^2 + \tfrac{5}{2}g(n-1) + 2\bigr) \cdot 2^{g(n)-2g(n-1)}$$

For $n \ge 2$, the right-hand side is upper bounded by

$$\bigl(2g(n-1)^2/4^{g(n-1)}\bigr) \cdot 2^{g(n)}.$$

For lower bounds, Wolfe and Fraser show that $g(n) \ge 2^{g(n-1)^{\alpha}}$ where $\alpha > .51$ and $\alpha \to 1$ as $n \to \infty$. For their proofs, they exploit knowledge of the join irreducibles of the day $n$ lattice mentioned in Section 3.

It would be of interest to tighten these bounds, particularly if doing so entailed describing the relationships between day $n$ and day $n+1$ in more detail. Is the middle level of each lattice the widest? Are the level sizes monotonic nondecreasing down to the middle level? (There are four levels with 5 games in the day 3 all-smalls.) Determine bounds on the number of all-smalls born by day $n$.

## 6. Further work

There are several other directions for further work besides those mentioned in the body of the survey.

While all of the above results were stated for short games (i.e., games born by day $n$ for $n < \omega$), proofs by induction imply similar results for $\mathcal{G}_\alpha$ where $\alpha$ is a transfinite ordinal [Siegel 2006]. However, Aaron Siegel's results concerning the all-small lattice do not generalize so easily, for it is not clear what ordinal multiples of $\uparrow$ should be.

Berlekamp (personal communication) has suggested other possible definitions for games born by day $n$, $\mathcal{G}_n$, depending on how one defines $\mathcal{G}_0$. The usual definition is 0-based, as $\mathcal{G}_0 = \{0\}$. Other natural definitions are integer-based (where $\mathcal{G}_0$ are integers) or number-based. While these two alternatives do not yield distributive lattices, perhaps there is still combinatorial structure worth investigating.

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