Probability, Geometry and Integrable Systems
MSRI Publications
Volume **55**, 2007

Homogenization of random Hamilton–Jacobi–Bellman Equations

S. R. SRINIVASA VARADHAN

ABSTRACT. We consider nonlinear parabolic equations of Hamilton–Jacobi–Bellman type. The Lagrangian is assumed to be convex, but with a spatial dependence which is stationary and random. Rescaling in space and time produces a similar equation with a rapidly varying spatial dependence and a small viscosity term. Motivated by corresponding results for the linear elliptic equation with small viscosity, we seek to find the limiting behavior of the solution of the Cauchy (final value) problem in terms of a homogenized problem, described by a convex function of the gradient of the solution. The main idea is to use the principle of dynamic programming to write a variational formula for the solution in terms of solutions of linear problems. We then show that asymptotically it is enough to restrict the optimization to a subclass, one for which the asymptotic behavior can be fully analyzed. The paper outlines these steps and refers to the recently published work of Kosygina, Rezakhanlou and the author for full details.

Homogenization is a theory about approximating solutions of a differential equation with rapidly varying coefficients by a solution of a constant coefficient differential equation of a similar nature. The simplest example of its kind is the solution u^ε of the equation

$$u^\varepsilon_t = \tfrac{1}{2} a\Big(\frac{x}{\varepsilon}\Big) u^\varepsilon_{xx}; \quad u^\varepsilon(0,x) = f(x)$$

on $[0,\infty]\times\mathbb{R}$. The function $a(\cdot)$ is assumed to be uniformly positive, continuous and periodic of period 1. The limit u of u^ε exists and solves the equation

$$u_t = \frac{\bar a}{2} u_{xx}; \quad u(0,x) = f(x)$$

where $\bar a$ is the harmonic mean

$$\bar a = \bigg(\int_0^1 \frac{dx}{a(x)}\bigg)^{-1}.$$

Although this is a result about solutions of PDE's it can be viewed as a limit theorem in probability. If we consider the Markov process $x(t)$ with generator

$$\tfrac{1}{2}a(x)\,D_x^2$$

starting from 0 at time 0, as $t \to \infty$ the limiting distribution of $y(t) = \frac{x(t)}{\sqrt{t}}$ is Gaussian with mean 0 and variance $\bar{a}$. The actual variance of $y(t)$ is

$$E\biggl[\frac{1}{t}\int_0^t a(x(s))\,ds\biggr].$$

The result on the convergence of u^ε to u is seen to follow from an ergodic theorem of the type

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(x(s))\,ds = \bar{a}.$$

From the theory of Markov processes one can see an ergodic theorem of this type with

$$\bar{a} = \int a(x)\phi(x)\,dx,$$

where $\phi(x)$ is the normalized invariant measure on [0, 1] with end points identified. This is seen to be

$$\phi(x) = \biggl(\int_0^1 \frac{dx}{a(x)}\biggr)^{-1}\frac{1}{a(x)},$$

so that

$$\bar{a} = \int_0^1 a(x)\phi(x)\,dx = \biggl(\int_0^1 \frac{dx}{a(x)}\biggr)^{-1}.$$

We can consider the situation where $a(x) = a(x,\omega)$ is a random process, stationary with respect to translations in x. We can formally consider a probability space (Ω, Σ, P), and an ergodic action τ_x of $\mathbb{R}$ on Ω. We also have a function $a(\omega)$ satisfying $0 < c \le a(\omega) \le C < \infty$. The stationary process $a(x,\omega)$ is given by $a(x,\omega) = a(\tau_x\omega)$. Now the solution u^ε of

$$u_t^\varepsilon(t,x,\omega) = \tfrac{1}{2}a(x,\omega)u_{xx}^\varepsilon(t,x,\omega); \quad u^\varepsilon(0,x,\omega) = f(x)$$

can be shown to converge again, in probability, to the nonrandom solution u of

$$u_t(t,x) = \frac{\bar{a}}{2}u_{xx}(t,x); \quad u^\varepsilon(0,x) = f(x)$$

with

$$\bar{a} = \biggl(\int \frac{1}{a(\omega)}dP\biggr)^{-1}.$$

This is also an ergodic theorem for

$$\frac{1}{t}\int_0^t a(\omega(s))\,ds,$$

but the actual Markov process $\omega(t)$ for which the ergodic theorem is proved is one that takes values in Ω with generator

$$L = \tfrac{1}{2}a(\omega)\boldsymbol{D}^2,$$

where $\boldsymbol{D}$ is the generator of the translation group τ_x on Ω. The invariant measure is seen to be

$$dQ = \frac{\bar{a}}{a(\omega)}dP,$$

where

$$\bar{a} = \left(\int \frac{1}{a(x)}dP\right)^{-1}.$$

We will try to adapt this type of approach to some nonlinear problems of Hamilton-Jacobi–Bellman type. One part of the work that we outline here was done jointly with Elena Kosygina and Fraydoun Rezakhanlou and has appeared in print [Kosygina et al. 2006], while another part, carried out with Kosygina, has been submitted for publication.

The problems we wish to consider are of the form

$$u_t^\varepsilon + \frac{\varepsilon}{2}\Delta u^\varepsilon + H\Big(\frac{x}{\varepsilon}, \nabla u^\varepsilon, \omega\Big) = 0; \quad u(T,x) = f(x)$$

for $[0,T]\times\mathbb{R}^d$. Here f is a continuous function with at most linear growth. (Ω, Σ, P) is a probability space on which $\mathbb{R}^d$ acts ergodically as measure preserving transformations τ_x. $H(0,p,\omega)$ is a function on $\mathbb{R}^d\times\Omega$ which is a convex function of p for every ω and $H(x,p,\omega) = H(0,p,\tau_x\omega)$. It satisfies some bounds and some additional regularity. The problem is to prove that $u^\varepsilon \to u$ as $\varepsilon \to 0$, where u is a solution of

$$u_t + \overline{H}(\nabla u) = 0; \quad u(T,x) = f(x)$$

for some convex function $\overline{H}(p)$ of p and determine it.

The analysis consists of several steps. We might as well assume $T = 1$ and concentrate on $u^\varepsilon(0,0,\omega)$. First we note that, by rescaling, the problem can be reduced to the behavior of

$$\lim_{t\to\infty}\frac{1}{t}u^t(0,0,\omega),$$

where u is the solution in $[0,t]\times\mathbb{R}^d$, of

$$u_s + \tfrac{1}{2}\Delta u + H(x,\nabla u,\omega); \quad u(t,x) = tf\Big(\frac{x}{t}\Big).$$

The second step is to use the principle of dynamic programming to write a variational formula for $u^t(s,x,\omega)$. Denote by $L(\tau_x\omega, q)$ the convex dual

$$L(x,q,\omega) = \sup_p \big(\langle p,q\rangle - H(x,p,\omega)\big)$$

Let $b(s,x)$ be a function $b:[0,t]\times\mathbb{R}^d \to \mathbb{R}^d$. Let $\mathcal{B}$ denote the space all such bounded functions. For each $b\in\mathcal{B}$, we consider the linear equation

$$v_s^b + \tfrac{1}{2}\Delta v^b + \langle b(s,x), \nabla v^b\rangle - L(\tau_x\omega, b(s,x)) = 0, v(t,x) = tf\Big(\frac{x}{t}\Big);$$

then the solution $u(s,x)$ is $\sup_b v^b(s,x)$. If we denote by Q^b the Markov process with generator

$$\mathcal{L}_s^b = \tfrac{1}{2}\Delta + \langle b(s,x), \nabla\rangle$$

starting from $(0,0)$, then

$$v^b(0,0,\omega) = E^{Q^b}\bigg[tf(\frac{x(t)}{t}) - \int_0^t L(x(s), b(s,x(s)),\omega)\,ds\bigg]$$

and

$$u = \sup_{b\in\mathcal{B}} v^b$$

The third step is to consider a subclass of $\mathcal{B}$ of the form $b(t,x) = c(\tau_x\omega)$ with $c:\Omega\to\mathbb{R}^d$ chosen from a reasonable class $\mathcal{C}$. The solution v^b with this choice of $b(t,x) = b(x) = c(\tau_x\omega)$ will be denoted by v^c. We will show that for our choice of $\mathcal{C}$, the limit

$$\lim_{t\to\infty}\frac{1}{t}v^c(0,0,\omega) = g(c)$$

will exist for every $c\in\mathcal{C}$. It then follows that

$$\liminf_{t\to\infty}\frac{1}{t}u^t(0,0) \geq \sup_{c\in\mathcal{C}} g(c).$$

Given c there is a Markov process $Q^{c,\omega}$ on Ω starting from ω with generator

$$\mathcal{A}_c = \tfrac{1}{2}\Delta + \langle c(\omega), \nabla\rangle.$$

Here ∇ is the infinitesimal generator of the $\mathbb{R}^d$ action $\{\tau_x\}$ and $\Delta = \nabla\cdot\nabla$. This process can be constructed by solving

$$dx(t) = c(\tau_{x(t)}\omega)\,dt + \beta(t); \quad x(0) = 0$$

Then one lifts it to Ω by defining $\omega(t) = \tau_{x(t)}\omega$. Such a process with generator $\mathcal{A}_c$ could have an invariant density P_c and it could (although it is unlikely) be mutually absolutely continuous with respect to P, having density Φ_c. Φ_c will be a weak solution of

$$\tfrac{1}{2}\Delta\Phi_c = \nabla\cdot c(\cdot)\,\Phi_c.$$

We can then expect

$$g(c) = f\left(\int c(\omega)\, dP_c \right) - \int L\big(\omega, c(\omega)\big)\, dP_c.$$

In general the existence of such a Φ for a given c is nearly impossible to prove. On the other hand for a given Φ finding a c is easy. For instance,

$$c = \frac{\nabla \Phi}{2\Phi}$$

will do. More generally one can have

$$c = \frac{\nabla \Phi}{2\Phi} + c',$$

so long as $\nabla \cdot c' \Phi = 0$. So pairs (c, Φ) such that

$$\tfrac{1}{2} \Delta \Phi_c = \nabla \cdot c(\,\cdot\,)\, \Phi_c$$

exist. Our class $\mathcal{C}$ will be those for which Φ exists. It is not hard to show, using the ergodicity of $\{\tau_x\}$ action, that Φ is unique for a given c when it exists and the Markov process with generator $\mathcal{A}_c$ is ergodic with $dP_c = \Phi_c dP$ as invariant measure. We will denote by $\mathcal{C}$ the class of pairs (c, Φ) satisfying the above relation. So we have a lower bound

$$\liminf_{t \to \infty} \frac{1}{t} u^c(0,0) \geq \sup_{m \in \mathbb{R}^d} [f(m) - I(m)]$$

where

$$I(m) = \inf_{\substack{c, \Phi : (c, \Phi) \in \mathcal{C}_0 \\ \int c \Phi\, dP = m}} \int L(c(\omega), \omega) \Phi\, dP$$

Now we turn to proving upper bounds. Fix $\theta \in \mathbb{R}^d$. If we had a "nice" test function $W(x, \omega)$ such that for almost all ω

$$|W(x, \omega) - \langle \theta, x \rangle| \leq o(|x|)$$

and

$$\tfrac{1}{2} \Delta W + H(x, \nabla W, \omega) \leq \lambda$$

Then, by convex duality with $\tilde{W} = W(x, \omega) - \lambda(s - t)$, we have

$$\tilde{W}_s + \tfrac{1}{2} \Delta \tilde{W} + \langle b(s, x), \nabla \tilde{W} \rangle - L(b(s, x), \omega) \leq 0.$$

If $\overline{H}(\theta)$ is defined as

$$\overline{H}(\theta) = \inf\{\lambda : W \text{ exists}\}$$

then under some control on the growth of L, it is not hard to deduce that with $f(x) = \langle\theta, x\rangle$,

$$\limsup_{t\to\infty} \frac{1}{t} u^t(0,0,\omega) \le \overline{H}(\theta)$$

If we can prove that

$$\overline{H}(\theta) = \sup_m [\langle\theta, m\rangle - I(m)],$$

we are done. We would match the upper and lower bounds. We reduce this to a minmax equals maxmin theorem.

$$\begin{aligned}
\sup_m[\langle\theta, m\rangle - I(m)] &= \sup_{(c,\Phi)\in\mathcal{C}} \int \big(\langle c(\omega), \theta\rangle - L(c(\omega),\omega)\big)\Phi\, dP \\
&= \sup_{(c,\Phi)} \inf_W \int \big(\langle c(\omega), \theta\rangle + \mathcal{A}_c W - L(c(\omega),\omega)\big)\Phi\, dP \\
&= \inf_W \sup_{(c,\Phi)} \int \big(\langle c(\omega), \theta\rangle + \mathcal{A}_c W - L(c(\omega),\omega)\big)\Phi\, dP \\
&= \inf_W \sup_\Phi \int \big(\tfrac{1}{2}\Delta W + H(\theta + \nabla W, \omega)\big)\Phi\, dP \\
&= \inf_W \sup_\omega \int \big(\tfrac{1}{2}\Delta W + H(\theta + \nabla W, \omega)\big)\Phi\, dP \\
&= \overline{H}(\theta).
\end{aligned}$$

While W may not exist, ∇W will exist. We can integrate on $\mathbb{R}^d$, then ergodic theorem will yield an estimate of the form $W(x) = o(|x|)$ and

$$\langle\theta, x\rangle + W(x)$$

will work as a test function. There are some technical details on the issues of growth and regularity. The details have appeared in [Kosygina et al. 2006] along with additional references. Similar results on the homogenization of random Hamilton–Jacobi–Bellman equations have been obtained by Lions and Souganidis [2005], using different methods.

Now we examine the time dependent case. If we replace $\mathbb{R}^d$ action by $\mathbb{R}^{d+1}$ action with (t, x) denoting time and space, then the stationary processes H and L are space time processes. The lower bound works more or less in the same manner. In addition to ∇ we now have D_t the derivative in the time direction. The $\omega(t)$ process is the space-time process. Its construction for a given c is slightly different. We start with $b(t, x) = c(\tau_{t,x}\omega)$ and construct a diffusion on $\mathbb{R}^d$ corresponding to the time dependent generator

$$\mathcal{A}_s^c = \tfrac{1}{2}\Delta + \langle b(s, x), \nabla\rangle$$

and then lift it by $\omega(s) = \tau_{s,x(s)}\omega$. The invariant densities are solutions of

$$-D_t\Phi + \tfrac{1}{2}\Delta\Phi = \nabla \cdot c\Phi.$$

The lower bound works the same way. But for obtaining the upper bound, a test function W has to be constructed that satisfies

$$W_t + \tfrac{1}{2}\Delta W + H(t, x, \nabla W, \omega) \le \overline{H}(\theta)$$

In the time independent case there was a lower bound on the growth of the convex function H that provided estimates on ∇W. Here one has to work much harder in order to control in some manner W_t. The details will appear in [Kosygina and Varadhan 2008].

References

[Kosygina and Varadhan 2008] E. Kosygina and S. R. S. Varadhan, "Homogenization of Hamilton–Jacobi–Bellman equations with respect to time-space shifts in a stationary ergodic medium", *Comm. Pure Appl. Math.* **61**:6 (2008).

[Kosygina et al. 2006] E. Kosygina, F. Rezakhanlou, and S. R. S. Varadhan, "Stochastic homogenization of Hamilton–Jacobi–Bellman equations", *Comm. Pure Appl. Math.* **59**:10 (2006), 1489–1521.

[Lions and Souganidis 2005] P.-L. Lions and P. E. Souganidis, "Homogenization of "viscous" Hamilton–Jacobi equations in stationary ergodic media", *Comm. Partial Differential Equations* **30**:1-3 (2005), 335–375.

S. R. SRINIVASA VARADHAN
COURANT INSTITUTE
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NY 10012
UNITED STATES
varadhan@cims.nyu.edu