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Two Proofs for Sylvester's Problem Using an Allowable Sequence of Permutations

HAGIT LAST

Abstract. The famous Sylvester's problem is: Given finitely many noncollinear points in the plane, do they always span a line that contains precisely two of the points? The answer is yes, as was first shown by Gallai in 1944. Since then, many other proofs and generalizations of the problem appeared. We present two new proofs of Gallai's result, using the powerful method of allowable sequences.

1. Introduction

Sylvester [1893] raised the following problem: Given finitely many noncollinear points in the plane, do they always span a simple line (that is, a line that contains precisely two of the points)? The answer is yes, as was first shown by Gallai [1944].

By duality, the former question is equivalent to the question: Given finitely many straight lines in the plane, not all passing through the same point, do they always determine a simple intersection point (a point that lies on precisely two of the lines)?

A natural generalization is to find a lower bound on the number of simple lines (or simple points, in the dual version). The dual version of this question can be generalized to pseudolines. The best lower bound [Csima and Sawyer 1993] states that an arrangement of n pseudolines in the plane determines at least $6n/13$ simple points. The conjecture [Borwein and Moser 1990] is that there are at least $n/2$ simple points for $n \neq 7, 13$. For the history of Sylvester's problem, with its many proofs and generalizations, see [Borwein and Moser 1990; Nilakantan 2005].

This paper presents two new proofs of Gallai's result using allowable sequences. A proof of Gallai's result using allowable sequences was given recently by Nilakantan [2005], but it differs from the two given here.

The notion of *allowable sequences* was introduced by Goodman and Pollack [1980]. It has proved to be a very effective tool in discrete and computational geometry; for a broad discussion see [Goodman and Pollack 1993]. Here is a short description of the notion.

Let S be a set of n points in the plane, let L be the set of the lines spanned by S, and let $\{k_1, k_2, \ldots, k_m\}$ be the m different slopes of the lines according to a fixed coordinate system. We choose a directed line l in the plane with a point P on it, such that l does not contain any point of S and is not orthogonal to any line in L.

Here is the construction of $\mathcal{A}_{l,P}(S)$, the allowable sequence of permutations of a point set S, according to the directed line l and the point P: We label the points of S according to their orthogonal projection on l and we get the first permutation $\pi_0 = 1, \ldots, n$. Let l rotate counterclockwise around P by $180°$ and look at the orthogonal projections of the labeled points of S on l as it rotates. A new permutation arises whenever l passes through a direction orthogonal to one of the slopes $k_1, k_2, \ldots, k_m$. It follows that along the course of this rotation, beside π_0, we will get m different permutations: $\pi_1, \ldots, \pi_m$. Define $\mathcal{A}_{l,P}(S) = \{\pi_0, \pi_1, \pi_2, \ldots, \pi_m\}$.

For each $1 \leq i \leq m$, whenever l passes through a direction orthogonal to k_i, the new permutation that arises differs from the previous one by reversing the order of the consecutive elements whose corresponding points of S lie on a line of slope k_i. Such reversed consecutive elements are called *a reversed substring.* If t lines in L have a slope equal to k_i, the permutation that corresponds to k_i has t disjoint reversed substrings. A reversed substring of length 2 is called *a simple switch.* A simple switch corresponds to a simple line.

Three important properties of $\mathcal{A}_{l,P}(S)$ are:

1. $\mathcal{A}_{l,P}(S)$ is a sequence of permutations of the elements $\{1, 2, \ldots, n\}$, where n is the cardinality of S.
2. The first permutation is $\pi_0 = 1, \ldots, n-1, n$, and the last is $\pi_m = n, n-1, \ldots, 1$. Here m is the number of different slopes of the lines spanned by S. If the points of S are not collinear, then $m > 1$ (actually $m \geq n-1$, as was proved in [Scott 1970]).
3. In the course of the sequence of permutations, every pair $i < j$ switches exactly once and so each permutation differs from the previous one by reversing at least one increasing substring. Only increasing substrings are reversed.

For example, if $\pi_i = 1, 7, 2, 4, 6, 3, 5$, then $N_1 = 1, 7$, $N_2 = 2, 4$, $N_3 = 2, 4, 6$, and $N_4 = 3, 5$ are its increasing substrings, and so π_{i+1} is obtained from π_i by reversing the order of one or more of these substrings.

For the convenience of writing the proofs in Section 2, we would like to assume that in each step only one increasing substring is reversed. We can arrange this by replacing each permutation that contains t reversed substrings by t permutations, as we reverse a single substring at a time. The length of the new sequence of permutations, $\mathcal{A}$, is the cardinality of L and satisfies the condition that each permutation differs from the previous one by reversing a single increasing substring.

2. The Proofs

Let S be a set of n noncollinear points in the plane. We will show the existence of a simple spanned line by proving that $\mathcal{A}$ contains a permutation with a simple switch. Assume, for a contradiction, that each reversed substring has length at least 3.

Since S is a set of noncollinear points, then $\mathcal{A}$ has length greater than 2, with $\pi_0 = 1, 2, \ldots, n$ and $\pi_m = n, n-1, \ldots, 1$ $(m > 1)$.

For $1 \leq r \leq m$, denote by J_r the reversed substring of π_r and denote by I_r the increasing substring of π_{r-1} which is reversed at π_r. J_r and I_r consist of the same set of elements, in J_r the elements are in decreasing order and in I_r they are in increasing order. For example, if $\pi_1 = 1, 2, 5, 4, 3$, $\pi_2 = 5, 2, 1, 4, 3$, then, $I_2 = 1, 2, 5$, $J_2 = 5, 2, 1$.

For $J_r = a_1, a_2, \ldots, a_{k-1}, a_k$, we will refer to $a_2, \ldots, a_{k-1}$ as its *internal elements*. By our assumption, every I_r as well as every J_r has an internal element.

Proof 1. We show that an internal element of a reversed substring cannot change its location before a simple switch occurs.

For every $0 \leq k \leq m$ and every element $1 \leq a \leq n$, denote by $T_k(a)$ the location of the element a in π_k. For example, $T_m(n-1) = 2$.

If $T_k(a) \neq T_{k-1}(a)$, we say that J_K *changed the location of the element* a. If $T_k(a) > T_{k-1}(a)$, we say that *a moves to the right at π_k*.

A reversed substring, J_r, is *centrally symmetric*, if it is symmetric around the middle of the permutation. For example, If $\pi_1 = 1, 2, 3, \underline{6, 5, 4}, 7, 8, 9$, then $J_1 = 6, 5, 4$ is centrally symmetric.

Let s be the smallest number such that J_s changes the location of an element which was an internal element in J_t for $t < s$. Such s must exist, otherwise, all internal elements of J_1 are already on their final positions at π_m. This means that J_1 is centrally symmetric. But then J_2 cannot be centrally symmetric and so its internal elements must later change their locations in order to be on their final positions at π_m.

Let a be an internal element of J_t with $t < s$, such that J_s changes the location of a. Without loss of generality, $T_s(a) > T_t(a)$. Since a moves to the right, there exist b, c such that a, b, c are consecutive elements of π_{s-1} and $a < b < c$. Since a is an internal element of J_t, there are d, e such that d, a, e are consecutive elements of π_t and $d > a > e$.

Let π_l, $t < l < s-1$, be the first permutation in which b is the right neighbor of a. Then there exist f, g such that a, b, f, g are consecutive elements of π_l and $a < b > f > g$. Since $T_{s-1}(a) = T_l(a)$, it follows that $T_{s-1}(c) = T_l(f)$. That means that before a moves to the right at π_s, f needs to change its location. But

f is an internal element in J_l and so, no J_d, $l < d < s$, can change the location of f (otherwise, it contradicts the definition of s). We conclude that such s cannot exist, which leads a contradiction. □

SECOND PROOF. A substring of three consecutive elements x, y, z in a permutation is called *a bad triplet* if $x < z$ but x, y, z are not in an increasing order.

Let π_l be the last permutation that contains a bad triplet x, y, z. Such π_l exists because π_1 has a bad triplet but π_m does not. For example, if π_1, π_m are $\pi_1 = 1, \underline{4, 3, 2}, 5, 6, \quad \pi_m = 6, 5, 4, 3, 2, 1$, then π_1 has two bad triplets $1, 4, 3$ and $3, 2, 5$. π_m is in decreasing order, so it contains no bad triplet.

To get a contradiction, we show here that at least one of the permutations that follows π_l contains a bad triplet.

Suppose that none of the permutations that follows π_l contains a bad triplet. Then either x or z (but not both) are elements of J_{l+1}. Assume that $x \in J_{l+1}$ (similar arguments can be used for the case $z \in J_{l+1}$).

We define the *closed interval* $[a, b]_d$ to be the part of the permutation π_d that contains the consecutive elements between a and b including a and b. Example, for $\pi_d : 6, 3, 2, 1, 5, 4 \quad [3, 5]_d = 3, 2, 1, 5$.

We now consider two cases:

Case 1*:* $x, y \in J_{l+1}$.

Then x is the right neighbor of y in J_{l+1}, and J_{l+1} contains at least one more element to the right of x. Let a be the rightmost element of J_{l+1} and b its left neighbor. Then $z > x \geq b > a$, from which follows that b, a, z are consecutive elements of π_{l+1} satisfying $b > a < z$ and $b < z$, which means that b, a, z is a bad triplet.

Case 2*:* $x \in J_{l+1}$, $y \notin J_{l+1}$.

Let $s = \min\{k \mid k > l+1 \text{ and } x \in J_k \text{ is not the leftmost element in } J_k\}$. Such s exists since $z > x$ and z, x are not yet reversed at π_{l+1}. Denote by c the left neighbor of x in J_s. Then x, c are consecutive elements of π_{s-1} and $x < c$.

Let $t = \max\{k \mid k < s \text{ and } x \in J_k\}$. Note that since x is an element of J_{l+1} and $l + 1 < s$, such t exists and satisfies $l + 1 \leq t < s$. Also, note that since x is the leftmost element of J_{l+1}, x is the leftmost element in J_t.

Let a, b be the two right neighbors of x in J_t. Then x, a, b are three consecutive elements of π_t and $x > a > b$.

Since $x \notin J_r$ for $t < r < s$, it follows that in order for c to be the right neighbor of x in π_{s-1}, c must switch with b first, and then with a, in permutations between t and s. So there exists r, $t < r < s$, such that $c, b \in J_r$ and there exists q, $r < q < s$, such that $c, a \in J_q$.

We claim that for every j satisfying $t \leq j < s$, $[x, b]_j$ contains no increasing substring of length greater than 2. Also, the three rightmost elements in $[x, b]_j$ are in decreasing order.

We will prove it by induction. For $j = t$ the claim holds. By the induction hypothesis, the three rightmost elements in $[x, b]_{j-1}$ are in decreasing order and

$I_j \not\subset [x,b]_{j-1}$. Since, in addition, $x \notin I_j$, it follows that if I_j contains elements of $[x,b]_{j-1}$, it must contain b only. If it does, the three rightmost elements of J_j are the three rightmost elements of $[x,b]_j$ and are in decreasing order.

Any increasing substring in $[x,b]_j$ can consist of only two elements, each of which belongs to a different reversed substring involving b. This completes the proof of the claim. By the definition of r, for every j satisfying $r \le j < s$ we have $c \in [x,b]_j$, but by the above claim, $I_j \not\subset [x,b]_{j-1}$, which implies that c cannot switch with a in a permutation that precedes π_s. So q as defined above cannot exist: a contradiction. □

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Hagit Last
Institute of Mathematics
The Hebrew University
91904 Jerusalem
Israel
hagitl@math.huji.ac.il