Combinatorial and Computational Geometry
MSRI Publications
Volume **52**, 2005

On the Size of Higher-Dimensional Triangulations

PETER BRASS

Abstract. I show that there are sets of n points in three dimensions, in general position, such that any triangulation of these points has only $O(n^{5/3})$ simplices. This is the first nontrivial upper bound on the MinMax triangulation problem posed by Edelsbrunner, Preparata and West in 1990: What is the minimum over all general-position point sets of the maximum size of any triangulation of that set? Similar bounds in higher dimensions are also given.

1. Introduction

In the plane, all triangulations of a set of points use the same number of triangles. This is a simple consequence of each triangle having an interior angle sum of π, and each interior point of the convex hull contributing an angle sum of 2π, which must be used up by the triangles.

Neither the constant size of triangulations nor the constant angle sum of simplices holds in higher dimensions. A classic example is the cube, which can be decomposed in two ways: into five simplices (cutting off alternate vertices) or into six simplices (which are even congruent; it is a well-known simple geometric puzzle to assemble six congruent simplices, copies of $\mathrm{conv}\big((000),(100),(010),(011)\big)$, into a cube).

For higher-dimensional cubes, the same problem was studied in a number of papers [Böhm 1989; Broadie and Cottle 1984; Haiman 1991; Hughes 1993; Hughes 1994; Lee 1985; Marshall 1998; Orden and Santos 2003; Sallee 1984; Smith 2000]. This suggest that one should be interested in the possible values of the numbers of simplices for arbitrary point sets.

It is well known that a triangulation of n points in d-dimensional space has size $\Omega(n)$ and $O(n^{\lceil d/2\rceil})$. The lower bound is obvious (each point must go somewhere); and, at least in three-dimensional space, as upper bound one can use that

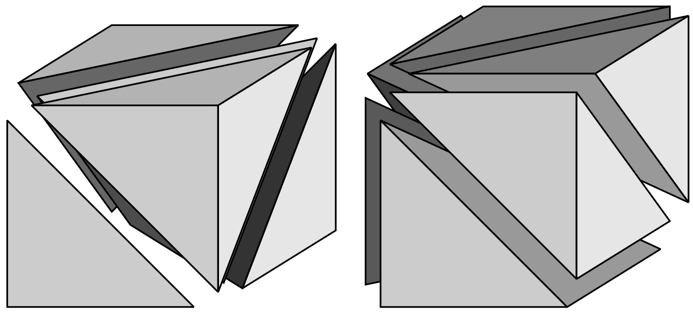

Figure 1. A cube can be triangulated with five or six simplices.

from each point the outer facets of the incident simplices can be viewed as faces of a starshaped polytope with at most $n-1$ vertices, which is combinatorially isomorphic to a convex polytope.

In more detail this problem was solved by Rothschild and Straus [1985], who showed that the minimum number of simplices in any triangulation of any full-dimensional set of n points in d-dimensional space is $n-d$. This is reached by gluing simplices together along faces, such that each additional simplex generates a new vertex, and all vertices are in convex position. Another method, without the general position, would be to place $n-d+1$ points on a line, and $d-1$ points off that line. They also showed that the maximum number of simplices in any triangulation of any full-dimensional set of n points in d-dimensional space is $\text{cyc\_poly}(n+1, d, d+1) - (d+1) = \Theta(n^{\lceil d/2 \rceil})$, where $\text{cyc\_poly}(n+1, d, d+1)$ is the number of d-faces of the $d+1$-dimensional cyclic polytope on $n+1$ vertices. This is a consequence of the upper bound theorem for simplicial d-spheres [Stanley 1983].

These were the maximum and minimum triangulation size, taken over all sets of n points in d-dimensional space. As a next step, it would be interesting to give bounds on the maximum and minimum triangulation size of a fixed set [Rothschild and Straus 1985, Problem 6.2]. For that we have to make some general-position assumption, no $d+1$ points collinear, otherwise there are always point sets for which there is only a unique triangulation. The questions are:

MaxMin Problem. *What is the smallest number $f_d^{\text{MaxMin}}(n)$, such that each set of n points in d-dimensional space, no $d+1$ collinear, has a triangulation with at most $f_d^{\text{MaxMin}}(n)$ simplices?*

MinMax Problem. *What is the largest number $f_d^{\text{MinMax}}(n)$, such that each set of n points in d-dimensional space, no $d+1$ collinear, has a triangulation with at least $f_d^{\text{MinMax}}(n)$ simplices?*

This problem was considered in three-dimensional space by Edelsbrunner, Preparata and West [Edelsbrunner et al. 1990], who showed that $f_3^{\text{MaxMin}}(n) \le 3n-11$, so every set of n point in general position in three-dimensional space has a small triangulation. They also gave some bounds, if additionally the number of points of the convex hull is given. Together with the lower bound of Sleator, Tarjan and Thurston [Sleator et al. 1988], who constructed a convex polyhedron which requires $2n-10$ simplices in any triangulation, this determines the exact minimum for point sets in convex position, and leaves only a linear-sized gap in general.

For higher dimensions, the vertices of a cyclic polytope give a lower bound for $f_d^{\mathrm{MaxMin}}(n)$, since in any triangulation of the cyclic polytope, each facet must be facet of some simplex, and each simplex has only $d+1$ facets. Together with the above-mentioned general upper bound of [Rothschild and Straus 1985] on any triangulation this shows

$$\Omega\big(\text{cyc\_poly}(n, d-1, d)\big) \le f_d^{\mathrm{MaxMin}}(n) \le O\big(\text{cyc\_poly}(n+1, d, d+1)\big),$$

so

$$\Omega\big(n^{\lfloor d/2 \rfloor}\big) \le f_d^{\mathrm{MaxMin}}(n) \le O\big(n^{\lceil d/2 \rceil}\big).$$

For the MinMax-Problem, the situation is much worse, only constant-factor improvements for the trivial lower and upper bounds are known [Edelsbrunner et al. 1990; Urrutia 2003], so $\Omega(n) \le f_3^{\mathrm{MinMax}}(n) \le O(n^2)$; and although some other problems raised in [Edelsbrunner et al. 1990] were solved [Bern 1993], no progress on the growth rate of $f_3^{\mathrm{MinMax}}(n)$ was made since then. It is the aim of this paper to prove the first nontrivial upper bound.

THEOREM 1. $f_3^{\mathrm{MinMax}}(n) = O(n^{5/3})$.

This follows from

LEMMA 2. *Any triangulation of a point set in three-dimensional space that arises by a small perturbation from the $n^{1/3} \times n^{1/3} \times n^{1/3}$ lattice cube contains at most $O(n^{5/3})$ simplices.*

This upper bound is probably not sharp even in that class of perturbed lattice cubes. It is easy to construct a perturbed lattice cube that allows a triangulation of size $\Omega(n^{4/3})$, and that is probably the true maximum in that class.

The same argument works also in higher dimensions, unfortunately the improvement over the general upper bound of $O(n^{\lceil d/2 \rceil})$ on the number of simplices in any d-dimensional triangulation is very small, especially if compared with the only known (trivial) lower bound $f_d^{\mathrm{MinMax}}(n) = \Omega(n)$.

THEOREM 3. $f_d^{\mathrm{MinMax}}(n) = O\left(n^{(1/d)+(d-1)\lceil d/2 \rceil/d}\right)$ *for fixed dimension d.*

The improvement in the exponent is thus

$$\frac{1}{d}\Big(\Big\lceil \frac{d}{2} \Big\rceil - 1\Big) \approx \frac{1}{2}.$$

2. The Proof

Let X_n be a set of n points, which is obtained from the lattice cube

$$X_n^* = \{(x_1, x_2, x_3) \mid x_i \in \{1, \dots, n^{1/3}\}\}$$

by a small perturbation. Any point $p \in X$ has a unique preimage $p^* \in X^*$ before the perturbation was applied, and any simplex $\{p_1, p_2, p_3, p_4\} \subset X$ has a preimage $\{p_1^*, p_2^*, p_3^*, p_4^*\} \subset X^*$, which is a possibly degenerate simplex (points coplanar or even collinear). Let $\mathcal{T}$ be the triangulation of X, then we partition $\mathcal{T} = \mathcal{T}_3 \cup \mathcal{T}_{\leq 2}$ by classifying the simplices $T \in \mathcal{T}$ according to the affine dimension of their preimage T^*; a simplex $T \in \mathcal{T}_3$ has a nondegenerate simplex T^* as preimage, a simplex $T \in \mathcal{T}_{\leq 2}$ has a coplanar, or even collinear, fourtuple T^* (degenerate simplex) as preimage.

We have less than $6n$ simplices in $\mathcal{T}_3$, since any nondegenerate simplex in X^* is a nondegenerate simplex with integer coordinates, so it has volume at least $\frac{1}{6}$; and the volume of $\operatorname{conv}(X^*)$ is less than n.

The preimages T^* of simplices $T \in \mathcal{T}_3$ together partition the cube $\operatorname{conv}(X^*)$ into nondegenerate simplices, and the vertices of these simplices are points of X^* so we can refine this partition to a triangulation $\mathcal{S}^*$ of X^*. Each face of a simplex T^*, $T \in \mathcal{T}_3$ of the partition is a union of faces of simplices from the triangulation $\mathcal{S}^*$. The triangulation $\mathcal{S}^*$ still contains at most $6n$ simplices.

The main problem is to bound $|\mathcal{T}_{\leq 2}|$, the number of almost-degenerate simplices in $\mathcal{T}$. Consider a simplex $T \in \mathcal{T}_{\leq 2}$, its preimage T^* is some coplanar fourtuple of points in X^*. Now T^* cannot intersect the interior of the preimage S^* of any of the full-dimensional simplices $S \in \mathcal{T}_3$. So each $T \in \mathcal{T}_{\leq 2}$ has a preimage T^* that is contained in the union of the faces of the S^*, $S \in \mathcal{T}_3$, so also in the union of faces of the S^*, $S^* \in \mathcal{S}^*$. Therefore each $T \in \mathcal{T}_{\leq 2}$ has a preimage T^* that is contained in a lattice plane of X^* spanned by a face of some S^* of the triangulation $\mathcal{S}^*$. Let $\{E_i\}_{i \in I}$ be the set of planes spanned by faces of simplices of the triangulation $S^* \in \mathcal{S}^*$, and let a_i be the number of simplices $S^* \in \mathcal{S}^*$ which have a face contained in the plane E_i. Since each of the $S^* \in \mathcal{S}^*$ contributes four faces, we have

$$\sum_{i \in I} a_i < 24n.$$

Since T^* is contained in the union of faces of simplices $S^* \in \mathcal{S}^*$, this holds also for the vertices of T^*; so they are either vertices of faces of the triangulation $\mathcal{S}^*$, or contained in the sides or relative interior of faces, which is not possible in a triangulation $\mathcal{S}^*$ of X^*. So each vertex of T^* is a vertex of some simplex S^*, and therefore the numbers b_i of points from $X^* \cap E_i$ that are vertices of T^* contained in E_i satisfies $\sum_{i \in I} b_i < 72n$. But also each b_i is at most $|X^* \cap E_i|$, so we have $b_i \leq n^{2/3}$ for each i. But these b_i points contained in E_i can generate only less than $O(b_i^2)$ simplices, since any set of b_i points can span at most $O(b_i^2)$

nonoverlapping simplices. So the total number of simplices in $\mathcal{T}_{\le 2}$ is less than $\sum_{i\in I} Cb_i^2$ for some C. Thus

$$|\mathcal{T}_{\le 2}| \le \max\left\{ \sum_{i\in I} Cb_i^2 \;\middle|\; \sum_{i\in I} b_i < 72n,\, 0 < b_i \le n^{2/3} \right\} = O(n^{5/3}).$$

The d-dimensional version is proved in exactly the same way: the point set $X_{n,d}$ is any perturbation of the $n^{1/d}\times\cdots\times n^{1/d}$-lattice cube. Any triangulation $\mathcal{T}_{n,d}$ of such a set will contain at most $O(n)$ simplices with a full-dimensional preimage in the unperturbed lattice $X^*_{n,d}$, since any nondegenerate simplex with integer vertices has a volume at least $\frac{1}{d!}$. All the remaining simplices of the triangulation are near-degenerate, they have preimages which are contained in the union of faces of the full-dimensional simplices. The full-dimensional preimages of simplices partition the cube into nondegenerate simplices with vertices from $X^*_{n,d}$, and we can refine this to a triangulation $\mathcal{S}^*_{n,d}$ of $X^*_{n,d}$ with $O(n)$ simplices. The faces of this triangulation span a set of affine lattice subspaces. Each near-degenerate simplex has a preimage in one of these subspaces, and each vertex of that near-degenerate simplex has a preimage that is in $\mathcal{S}^*_{n,d}$ vertex of a simplex with a face that spans that affine subspace. The total number of pairs of vertices and incident faces in $\mathcal{S}^*_{n,d}$ is $O(n)$ and each of these pairs belongs to an affine lattice subspace, and can belong to the preimages of near-degenerate simplices only in that subspace. We sum now over all such subspaces, and count each point only for those subspaces where it is vertex with an incident face that spans the subspace. A subspace s that contains b_s points can contain only $O(b_s^{\lceil d/2\rceil})$ preimages of near-degenerate simplices, since that is the maximum number of simplices that these b_s points can span. And each subspace contains at most $n^{(d-1)/d}$ points, since that is the maximum intersection of a proper affine subspace with the lattice cube. We now consider this just as an abstract optimization problem for the variables b_s, and get an upper bound of

$$\max\left\{ \sum_{s} O(b_s^{\lceil d/2\rceil}) \;\middle|\; \sum_{s} b_s = O(n),\, 0 < b_s \le n^{(d-1)/d} \right\}.$$

This maximum is again reached if each nonvanishing b_s is as large as possible, so $b_s = n^{(d-1)/d}$ for $O(n^{1/d})$ variables b_s, which is the claimed bound.

3. Related Problems

The most important problem would be to get a nontrivial lower bound for $f_d^{\mathrm{MinMax}}(n)$. It is still possible that there are point sets which allow only linear-sized triangulations. Perhaps it might help to compute some exact values and extremal configurations for small n; the first nontrivial values seem to be

$$f_3^{\mathrm{MinMax}}(5) = 3 \quad\text{and}\quad f_3^{\mathrm{MinMax}}(6) = 5,$$

both realized by points in convex position.

A good lower bound on $f_d^{\text{MinMax}}(n)$ would also be interesting since it would imply an upper bound for the d-dimensional Heilbronn triangle problem. Let $g_d^{\text{MinVol}}(n)$ be the maximum over all choices of n points from the unit cube of the minimum volume of a simplex spanned by this set, then

$$g_d^{\text{MinVol}}(n) \le \frac{1}{f_d^{\text{MinMax}}(n)}.$$

For $d \ge 3$, the best upper bound we have on $g_d^{\text{MinVol}}(n)$ is only slightly better than the trivial bound [Brass 2005]; for lower bounds see [Barequet 2001; Lefmann 2000].

It should be possible to determine the exact function for $f_3^{\text{MaxMin}}(n)$, or at least the right multiplicative constant.

The problem of triangulating the d-cube with minimal number of simplices was already mentioned in the beginning. It does not quite fall in the model here, since the vertices of the cube are not in general position. The *maximum* number of simplices in any triangulation of the d-cube are $d!$, by the volume argument used above, and this number can be reached easily. The *minimum* number of simplices is known to be between

$$\frac{1}{2\sqrt{d+1}}\left(\frac{6}{d+1}\right)^{d/2} d! \qquad \text{and} \qquad (0.816)^d d!$$

(see [Smith 2000] and [Orden and Santos 2003], respectively); so the gap between upper an lower bound is still enormous, of order $2^{\Theta(d \log d)}$.

References

[Barequet 2001] G. Barequet, "A lower bound for Heilbronn's triangle problem in d dimensions", *SIAM J. Discrete Math.* **14**:2 (2001), 230–236.

[Bern 1993] M. Bern, "Compatible tetrahedralizations", pp. 281–288 in *Proceedings of the Ninth Annual Symposium on Computational Geometry* (San Diego, 1993), ACM Press, New York, 1993.

[Böhm 1989] J. Böhm, "Some remarks on triangulating a d-cube", *Beiträge Algebra Geom.* no. 29 (1989), 195–218.

[Brass 2005] P. Brass, "An upper bound for the d-dimensional Heilbronn triangle problem", *SIAM J. Discrete Math* (2005). To appear.

[Broadie and Cottle 1984] M. N. Broadie and R. W. Cottle, "A note on triangulating the 5-cube", *Discrete Math.* **52**:1 (1984), 39–49.

[Edelsbrunner et al. 1990] H. Edelsbrunner, F. P. Preparata, and D. B. West, "Tetrahedrizing point sets in three dimensions", *J. Symbolic Comput.* **10**:3-4 (1990), 335–347.

[Haiman 1991] M. Haiman, "A simple and relatively efficient triangulation of the n-cube", *Discrete Comput. Geom.* **6**:4 (1991), 287–289.

[Hughes 1993] R. B. Hughes, "Minimum-cardinality triangulations of the d-cube for $d = 5$ and $d = 6$", *Discrete Math.* **118**:1-3 (1993), 75–118.

[Hughes 1994] R. B. Hughes, "Lower bounds on cube simplexity", *Discrete Math.* **133**:1-3 (1994), 123–138.

[Lee 1985] C. W. Lee, "Triangulating the d-cube", pp. 205–211 in *Discrete geometry and convexity* (New York, 1982), edited by J. E. Goodman et al., Ann. New York Acad. Sci. **440**, New York Acad. Sci., New York, 1985.

[Lefmann 2000] H. Lefmann, "On Heilbronn's problem in higher dimension", pp. 60–64 in *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms* (San Francisco, 2000), ACM, New York, 2000.

[Marshall 1998] T. H. Marshall, "Volume formulae for regular hyperbolic cubes", *Conform. Geom. Dyn.* **2** (1998), 25–28.

[Orden and Santos 2003] D. Orden and F. Santos, "Asymptotically efficient triangulations of the d-cube", *Discrete Comput. Geom.* **30**:4 (2003), 509–528.

[Rothschild and Straus 1985] B. L. Rothschild and E. G. Straus, "On triangulations of the convex hull of n points", *Combinatorica* **5**:2 (1985), 167–179.

[Sallee 1984] J. F. Sallee, "The middle-cut triangulations of the n-cube", *SIAM J. Algebraic Discrete Methods* **5**:3 (1984), 407–419.

[Sleator et al. 1988] D. D. Sleator, R. E. Tarjan, and W. P. Thurston, "Rotation distance, triangulations, and hyperbolic geometry", *J. Amer. Math. Soc.* **1**:3 (1988), 647–681.

[Smith 2000] W. D. Smith, "A lower bound for the simplexity of the n-cube via hyperbolic volumes", *European J. Combin.* **21**:1 (2000), 131–137.

[Stanley 1983] R. P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics **41**, Birkhäuser, Boston, 1983.

[Urrutia 2003] J. Urrutia, "Coloraciones, tetraedralizaciones, y tetraedros vacios en coloraciones de conjuntos de puntos en $\mathbb{R}^3$", pp. 95–100 in *Proc. X Encuentros de Geometria Computacional* (Sevilla, 2003), edited by C. Grima et al., Universidad de Sevilla, 2003.

Peter Brass
City College of New York, CUNY
Department of Computer Science
138th Street at Convent Ave.
New York, NY 10031
peter@cs.ccny.cuny.edu